

Meissel's theorem in additive arithmetical semigroups

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Abstract

We show how to control the error term in Mertens' formula and related theorems in the context of additive arithmetical semigroups and carry over an old related result of Meissel.

2000 Mathematics Subject Classification: 11N80, 11A41, 11N05

Keywords: arithmetical semigroups, Mertens' theorem

1 Introduction

Many algebraic and combinatorial structures may be viewed as multisets of indecomposable components. As a general framework, John Knopfmacher defined the notion of *additive arithmetical semigroup* in his influential books [3] and [4].

An additive arithmetical semigroup is a free commutative monoid G with non-empty set of generators P , equipped with a non-negative integer-valued degree mapping ∂ such that

$$\partial(ab) = \partial(a) + \partial(b)$$

for all $a, b \in G$, and such that

$$G(n) := \#\{a \in G; \partial(a) = n\} < \infty$$

for all $n \in \mathbb{N}_0$. Equivalently, we may require that

$$P(n) := \#\{p \in P; \partial(p) = n\}$$

be finite for all n . In particular, G is countable.

We denote the generating function by

$$Z(y) := \sum_{n=0}^{\infty} G(n)y^n.$$

Its radius of convergence ρ must satisfy $0 \leq \rho \leq 1$ since $\#G = \sum_n G(n)$ is infinite. Any sensible classification must treat the cases $\rho = 0$ and $\rho = 1$ separately; in this paper, we restrict our attention to the case $0 < \rho < 1$. Following the traditional notation by Knopfmacher, we set

$$q := 1/\rho.$$

In the past decades, many authors have investigated the connection between $P(n)$ and $G(n)$. A usual hypothesis is that

$$\frac{G(n)}{q^n} = A + r(n) \tag{1}$$

with some $A > 0$ and

$$r(n) \rightarrow 0$$

sufficiently fast; we term this condition axiom \mathcal{A} (with error term $r(n)$). A desired result would be to show that

$$\lambda_n := \frac{nP(n)}{q^n} \tag{2}$$

converges to 1, in analogy to the prime number theorem; however, this does not necessarily hold even if $r(n) \rightarrow 0$ exponentially fast, see [1]. Wen-Bin Zhang has proved that if $\sum_{n=0}^{\infty} \sup_{k \geq n} |r(k)|$ converges, then λ_n is bounded. On the other hand, in a separate paper ([2]), Indlekofer and I have shown that examples exist where $r_n = O(\ln(n)^2/n)$ and λ_n is unbounded.

However, it is possible to show that the Cesaro limit of λ_n exists; more precisely, Zhang and Knopfmacher showed ([5], Section 3.3):

Proposition 1.1. *Suppose that $\sum_{n=0}^{\infty} \sup_{k \geq n} |r(k)|$ converges. Then*

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$$\sum_{k=1}^n \lambda_k = n + O(1). \tag{3}$$

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$$\sum_{k=1}^n \frac{P(k)}{q^k} = \ln(n) + C_1 + O(1/n) \tag{4}$$

for some constant C_1 .

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$$\prod_{k=1}^n (1 - 1/q^k)^{P(k)} = \frac{C_2}{n} + O(1/n^2) \quad (5)$$

for some constant C_2 .

In [2], Indlekofer and I have shown that Zhang's estimates hold under a much weaker condition, at the price of obtaining no error term:

Proposition 1.2. *Suppose that*

$$Z(y) = \frac{H(y)}{1 - qy} \quad (6)$$

and $H(y) \rightarrow A > 0$ as $y \rightarrow 1/q$. Then

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$$\sum_{\substack{p \in P, k \in \mathbb{N} \\ \partial(p^k) \leq n}} \frac{1}{kq^{k\partial(p)}} = \ln(n) + \gamma + \ln(A) + o(1) \quad (7)$$

where γ denotes Euler's constant.

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$$\sum_{k=1}^n \frac{P(k)}{k} = \ln(n) + \gamma + \ln(A) - C_M + o(1) \quad (8)$$

where

$$C_M = \sum_{k=1}^n P(k) \left(\frac{1}{q^k} - \ln \left(1 - \frac{1}{q^k} \right) \right).$$

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$$\sum_{k=1}^n \lambda_k = n + o(n).$$

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$$\prod_{k=1}^n \frac{1}{1 - 1/q^k} = A \exp(\gamma)n + o(n). \quad (9)$$

In particular, this gives us the values of C_1 and C_2 . Since the proof is based on a Tauberian theorem, it is difficult to turn this into a theorem with explicit error terms.

2 Mertens-type estimates error terms

In this section, it will become apparent how the trade-off between hypotheses and error terms in Propositions 1.1 and 1.2 can be controlled.

Lemma 2.1. *Suppose axiom \mathcal{A} . Then*

$$\sum_{k \in \mathbb{N}} kP(k) \frac{G(n-k)}{G(n)} = n - C_3 + o(1),$$

where

$$\begin{aligned} C_3 &:= \sum_{k=1}^{\infty} kP(k) \sum_{j=2}^{\infty} q^{-jk} \\ &= \sum_{k=1}^{\infty} kP(k) \frac{1}{q^k(q^k - 1)}. \end{aligned}$$

Proof. For all $n \in \mathbb{N}$, we have that

$$\begin{aligned} nG(n) &= \sum_{a \in G_n} \partial(a) \\ &= \sum_{a \in G_n} \sum_{p^j | a} \partial(p) \\ &= \sum_p \partial(p) \sum_{j=1}^{\infty} G(n - j\partial(p)) \\ &= \sum_{k=1}^{\infty} P(k)k \sum_{j=1}^{\infty} G(n - jk). \end{aligned}$$

If n is sufficiently large, we may divide both sides by $G(n)$, and axiom \mathcal{A} implies

$$\sum_{k=1}^{\infty} P(k)k \sum_{j=2}^{\infty} \frac{G(n-jk)}{G(n)} \rightarrow C_3$$

as $n \rightarrow \infty$ (note that the inner sum is actually finite), such that the lemma follows. \square

Theorem 2.2. *Suppose that λ_n is bounded. Let $r(n) = O(f(n))$ for a continuous nonincreasing function f with $\lim_{x \rightarrow \infty} f(x) = 0$; let $F(x) := \int_1^x f(t)dt$. Suppose that $\int_1^{\infty} F(x)/x^2$ converges. Then*

- (3) holds with error term $O(F(n))$.

- (4) holds with error term $O(\int_n^\infty F(x)/x^2 dx)$.

Proof. First note that $xf(x) = O(F(x))$. In view of the previous theorem, we have to estimate

$$\sum_{k=1}^n kP(k) \left(\frac{G(n-k)}{G(n)} - q^{-k} \right) = \sum_{k=1}^n \lambda_k \frac{r(n-k) - r(n)}{A + r(n)}. \quad (10)$$

such that our first assertion follows by reordering the sum $\sum_k r(n-k) = \sum_k r(k)$ and comparing it to an integral.

The second assertion follows by partial summation: Let $S(t) := \sum_{k=1}^{\lfloor t \rfloor} \lambda_k$. Let $s(t)$ defined by $S(t) = t + s(t)$; then $s(t) = O(F(t))$ by the first part. Let

$$I := \int_1^\infty \frac{s(t)}{t^2} dt.$$

Then

$$\begin{aligned} \sum_{k=1}^n \lambda_k \cdot \frac{1}{k} &= S(n) \cdot \frac{1}{n} + \int_1^n S(t) \frac{1}{t^2} dt \\ &= (n + s(n)) \cdot \frac{1}{n} + \int_1^n \frac{1}{t} dt + \int_1^n \frac{s(t)}{t^2} dt \\ &= 1 + O(F(n)/n) + \ln(n) + I + O\left(\int_n^\infty \frac{F(t)}{t^2} dt\right) \end{aligned}$$

Since $-F(n)/n = \int_n^\infty f(x)/x dx - \int_n^\infty F(x)/x^2$, it follows that the error term is $O(\int_n^\infty F(x)/x^2 dx)$. (The main terms are, of course, those of (8), such that $I + 1 = \gamma + \ln(A) - C_M$.) \square

It follows that Theorem 1.1 can be improved slightly.

Corollary 2.3. *The assertions of Theorem 1.1 hold under the weaker hypotheses $P(n) = O(q^n/n)$, that $r(n) = O(1/n)$, and that $\sum_{n=1}^\infty |r(n)|$ converges.*

Proof. From our assumptions, it is clear that the right hand side of (10) is bounded in this case. The second assertion follows by partial summation as before (this time with $F(x)$ replaced by 1), and the third one by exponentiation of the second and the definition of C_M . \square

3 A sum considered by Meissel

The following, nearly forgotten result predates Mertens' 1874 paper [7] containing the theorem named after him. See [6] for a proof and a full account of the historical background.

Proposition 3.1 (Meissel 1866). *For $\alpha \rightarrow 0+$,*

$$\sum_{p \in \mathbb{P}} \frac{1}{p(\ln(p))^\alpha} = \frac{1}{\alpha} + C + O(\alpha),$$

where

$$C = \lim_{n \rightarrow \infty} \left(\sum_{p \leq n} \frac{1}{p} - \ln(\ln(n)) \right).$$

We show that this result, too, carries over to the case of additive arithmetical semigroups:

Theorem 3.2. *Suppose $H(y) \rightarrow A$ for $y \rightarrow 1/q$. Let $C_1 := \ln(A) + \gamma - C_M$. Let $\alpha > 0$. Let*

$$S(x) := \sum_{k=1}^{\lfloor x \rfloor} \frac{P(k)}{q^k}$$

and $s(x) := S(x) - \ln(x) - C_1$. Let

$$J(\alpha) := \int_1^\infty \frac{s(x)}{x^{\alpha+1}} dx \tag{11}$$

Then

$$\sum_{k=1}^\infty \frac{P(k)}{q^k k^\alpha} = 1/\alpha + C_1 + \alpha J(\alpha)$$

Proof. By partial summation.

$$\begin{aligned} \sum_{k=1}^N \frac{P(k)}{q^k k^\alpha} &= S(N) \frac{1}{N^\alpha} + \int_1^N S(x) \frac{\alpha}{x^{\alpha+1}} dx \\ &= \frac{\ln(N) + C_1 + s(N)}{N^\alpha} + \alpha \int_1^N \frac{\ln(x) + C_1 + s(x)}{x^{\alpha+1}} dx. \end{aligned}$$

Letting $N \rightarrow \infty$, the first summand on the right vanishes, and the summands of the integral are

$$\int_1^\infty \frac{\ln(x)}{x^{\alpha+1}} dx = \frac{1}{\alpha^2},$$

$$\int_1^\infty \frac{C_1}{x^{\alpha+1}} = \frac{C_1}{\alpha},$$

and (11), from which our result follows. \square

To arrive at the same result as Meissel, we need $J(0)$ to exist. By Theorem 2.2, it suffices if $\int_t^\infty F(x)/x^2 = O(1/\ln(t)^{1+\epsilon})$ holds. This finally gives us Meissel's theorem for additive arithmetical semigroups:

Corollary 3.3. *If axiom \mathcal{A} holds with error term $r(n) = O(1/\ln(n)^{2+\epsilon})$ and $P(n) = O(q^n/n)$, then*

$$\sum_{k=1}^{\infty} \frac{P(k)}{q^k k^\alpha} = 1/\alpha + C_1 + O(\alpha)$$

as $\alpha \rightarrow 0$.

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