NON-ISOMORPHIC DERIVED-EQUIVALENT TUBULAR CURVES AND THEIR ASSOCIATED TUBULAR ALGEBRAS

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Dedicated to Professor Helmut Lenzing on the occasion of his 60th birthday

ABSTRACT. Let $\mathbb{R}$ be the field of real numbers. We construct two tubular canonical $\mathbb{R}$-algebras (in the sense of Ringel/Crawley-Boevey [22]) which are neither isomorphic nor dual to each other, but which are tilting-equivalent. This relates to the fact that each of these algebras admits two distinct isomorphism classes of separating tubular families of standard stable tubes. The results are derived from the existence of two non-isomorphic tubular exceptional curves (in the sense of [15]) over $\mathbb{R}$ which are derived-equivalent, one having a commutative, the other a non-commutative function field. Furthermore, we classify all generic modules over such tubular algebras.

1. INTRODUCTION

In [1] Atiyah classified the vector bundles over an elliptic curve over an algebraically closed field proving that the indecomposable objects in the category $\mathrm{coh}(\mathbb{T})$ of coherent sheaves over an elliptic curve $\mathbb{T}$ (= curve of genus one) form a rational family of one-parameter families which are parametrized by $\mathbb{T}$; moreover all of these one-parameter families are isomorphic (for a fixed $\mathbb{T}$).

In [21] Ringel introduced the tubular (canonical) algebras over an algebraically closed field k and showed a related result that the indecomposable finite-dimensional modules over such an algebra can be classified basically by a rational family of pairwise isomorphic one-parameter families (so-called tubular families), in this case parametrized by the projective line $\mathbb{P}^1(k)$.

The connection between these results is given by the theory of weighted projective lines introduced by Geigle and Lenzing [6], and the tubular/elliptic situation is treated in detail by Meltzer and Lenzing [17], where the notion of tubular mutations plays the key role for classification (see [19, 17]). By definition the tubular weighted projective lines are just the weighted projective lines of virtual genus one, and thus are the weighted analogues of the elliptic curves.

In the present article we show that new effects occur if the base field is not algebraically closed. We show the existence of two non-commutative curves $\mathbb{X}_1$ and $\mathbb{X}_2$, each equipped with a tilting bundle T_i $(i = 1, 2)$ such that:

- The function field of $\mathbb{X}_1$ (resp. $\mathbb{X}_2$) is commutative (resp. non-commutative). In particular, $\mathbb{X}_1$ and $\mathbb{X}_2$ are non-isomorphic.
- $\Lambda_1 = \mathrm{End}(T_1)$ and $\Lambda_2 = \mathrm{End}(T_2)$ are tubular canonical algebras which are tilting-equivalent, but neither isomorphic nor anti-isomorphic. In particular, Λ_1 and Λ_2, hence $\mathbb{X}_1$ and $\mathbb{X}_2$ are derived-equivalent.

- The tubular families for each of $\operatorname{coh}(\mathbb{X}_1)$ and $\operatorname{coh}(\mathbb{X}_2)$ (resp. $\operatorname{mod}(\Lambda_1)$ and $\operatorname{mod}(\Lambda_2)$) fall into two classes of non-isomorphic tubular families, indexed by $\mathbb{X}_1$ and $\mathbb{X}_2$, respectively.

The results give a counterexample to a question of Bondal and Orlov ([3]), here in the more general setting of non-commutative curves over non-algebraically closed field.

Recall that a Λ_1-module is called *generic* [4], if it is indecomposable and of infinite dimension over the base field, but of finite endolength (that is, of finite length over its endomorphism ring). As application of our results we show that there are generic modules over Λ_1 (resp. Λ_2) with non-isomorphic endomorphism skew-fields, which is also different from the algebraically closed case [14].

Some results of this article are part of the author's doctoral thesis [10]. The author takes this opportunity to thank his graduate adviser, Professor Helmut Lenzing.

2. Statement of results

In this section we state the main results which will be proved in the following sections.

Throughout this article let $k = \mathbb{R}$ be the field of real numbers. Denote by $\mathbb{C}$ and $\mathbb{H}$ the field of complex numbers and the skew-field of real quaternions, respectively. Let Λ be a canonical k-algebra in the sense of Ringel/Crawley-Boevey [22]. We call Λ *tubular* if the radical of the Grothendieck group $\mathrm{K}_0(\Lambda)$ of $\operatorname{mod}(\Lambda)$ is a finitely generated free abelian group of rank two (see [13, 16] for characterizations).

Λ is defined by an underlying bimodule $M = {}_F M_G$ where F and G are finite dimensional skew-fields over k and M is a tame bimodule, which means that $(\dim_F M) \cdot (\dim M_G) = 4$, and k is acting centrally on F, G and M. The pair $(\dim_F M, \dim M_G)$ is called the *type* of M. We say that M or Λ has *numerical type* ε, where ε is defined to be 1 if $(\dim_F M, \dim M_G) = (2, 2)$, and it is 2 if $(\dim_F M, \dim M_G) \in \{(1, 4), (4, 1)\}$. If the canonical algebras Λ_1 and Λ_2 have different numerical type, Λ_1 is neither isomorphic to Λ_2 nor to Λ_2^{op}.

Denote by $\mathbb{X}_1$ the graded projective spectrum of the graded factorial algebra $\mathbb{R}[X, Y, Z]/(X^2 + Y^4 + Z^4)$ with grading group given by the abelian group with generators $\deg X$, $\deg Y$, $\deg Z$ and relations $\deg X = 2 \cdot \deg Y = 2 \cdot \deg Z$.

Theorem 1. *There are tubular canonical $\mathbb{R}$-algebras Λ_1 and Λ_2 which can be realized as endomorphism rings of tilting bundles in the category of graded coherent sheaves* $\operatorname{coh}(\mathbb{X}_1)$ *such that*

- *Λ_1 has numerical type $\varepsilon = 2$ and Λ_2 has numerical type $\varepsilon = 1$;*
- *Λ_1 and Λ_2 are tilting-equivalent.*

If Λ is (tubular) canonical then it admits a separating tubular family whose additive closure we denote by $\operatorname{mod}_0(\Lambda)$. In the tubular case such a separating tubular family is not uniquely determined; actually, there is a rational family of separating tubular families in $\operatorname{mod}(\Lambda)$. The category $\operatorname{mod}_0(\Lambda)$ is a coproduct $\coprod_{x \in \mathbb{X}} \mathcal{U}_x$ of connected uniserial categories $\mathcal{U}_x$ ("tubes"), where $\mathbb{X}$ is an index set.

In [16] it is shown that to such a tubular family $\operatorname{mod}_0(\Lambda)$ one can associate in a natural way a connected small abelian category $\mathcal{H} = \mathcal{H}(\Lambda, \operatorname{mod}_0(\Lambda))$ with the following properties:

- $\mathcal{H}$ is hereditary with no non-zero projectives;

- $\mathcal{H}$ is noetherian and locally finite (that is, Hom and Ext spaces are finite-dimensional over k);
- $\mathcal{H}$ admits a (torsion-free) tilting object whose endomorphism ring is Λ;
- If $\mathcal{H}_0$ denotes the full subcategory of $\mathcal{H}$ of objects of finite length then $\mathcal{H}_0 = \operatorname{mod}_0(\Lambda)$.

Since $\mathcal{H}$ shares the abstract properties with a category of coherent sheaves over a non-singular projective curve $\mathbb{X}$ it is also denoted by $\operatorname{coh}(\mathbb{X})$, and $\mathbb{X}$ (together with $\operatorname{coh}(\mathbb{X})$) is called an *exceptional curve* [15]. Tilting theory implies that the categories $\operatorname{mod}(\Lambda)$ and $\operatorname{coh}(\mathbb{X})$ are derived-equivalent. The exceptional curve $\mathbb{X}$ is called *tubular*, if Λ is tubular, or equivalently, if the virtual genus (see [15]) of $\mathbb{X}$ is one. We denote by $\mathrm{D}^b(\mathbb{X})$ the (bounded) derived category of $\operatorname{coh}(\mathbb{X})$. Two exceptional curves $\mathbb{X}_1$ and $\mathbb{X}_2$ are called *isomorphic* (*derived-equivalent*, respectively) if the corresponding categories $\operatorname{coh}(\mathbb{X}_1)$ and $\operatorname{coh}(\mathbb{X}_2)$ are equivalent (derived-equivalent, respectively). If $\mathbb{X}$ is an exceptional curve, then let $\operatorname{coh}_0(\mathbb{X}) = \mathcal{H}_0$. The quotient category $\frac{\operatorname{coh}(\mathbb{X})}{\operatorname{coh}_0(\mathbb{X})}$ is equivalent to a module category $\operatorname{mod}(D)$, where D is a skew-field (compare [2, Cor. 4.7]), called the *function field* of $\mathbb{X}$. For each sheaf $\mathcal{F} \in \operatorname{coh}(\mathbb{X})$ the *rank* $\operatorname{rk}\mathcal{F}$ is defined as the dimension of the induced vector space over D.

Theorem 2. *There are two non-isomorphic tubular exceptional curves $\mathbb{X}_1$ and $\mathbb{X}_2$ over $\mathbb{R}$ which are derived-equivalent and such that the function field of $\mathbb{X}_1$ is commutative and the function field of $\mathbb{X}_2$ is non-commutative. For $\mathbb{X}_1$ we can take the curve from Theorem 1.*

Let $\mathbb{X}$ be a tubular exceptional curve. As in [6] we define the slope as the quotient $\mu = \frac{\deg}{\operatorname{rk}}$, and define *stability* and *semi-stability* with respect to μ (compare [13] for the definition of the degree deg in this general setting). Since $\mathbb{X}$ is tubular, each indecomposable sheaf is semi-stable (see [6, Prop. 5.5]). For $q \in \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ we denote by $\mathcal{H}^{(q)}$ the full subcategory of $\mathcal{H} = \operatorname{coh}(\mathbb{X})$ consisting of the semi-stable sheaves of slope q and the zero sheaf. We call the categories $\mathcal{H}^{(q)}$ *slope categories*. Note that $\mathcal{H}$ is the additive closure of the slope categories.

Theorem 3. *Let $\mathbb{X}_1$ be the tubular exceptional curve from Theorem 1, let $\mathcal{H} = \operatorname{coh}(\mathbb{X}_1)$. Then each slope category $\mathcal{H}^{(q)}$ is non-trivial, and the group $\operatorname{Aut}(\mathrm{D}^b(\mathbb{X}_1))$ of auto-equivalences of $\mathrm{D}^b(\mathbb{X}_1)$ acts with precisely two orbits on the set of slope categories $\mathcal{H}^{(q)}$ ($q \in \overline{\mathbb{Q}}$). Moreover, slope categories belonging to different orbits are not equivalent.*

As an application we will show that there is a tubular canonical algebra over $\mathbb{R}$ which admits two isoclasses of separating tubular families of stable tubes. Let $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} = \mathbf{ij}$ denote the usual $\mathbb{R}$-basis of $\mathbb{H}$, we put $\mathbb{C} = \mathbb{R}\mathbf{1} \oplus \mathbb{R}\mathbf{i}$, $\mathbb{C}' = \mathbb{R}\mathbf{1} \oplus \mathbb{R}\mathbf{j}$.

Denote by $[1,\infty] \vee]{-\infty},0]$ the ordered subset of the lexicographically ordered set $\mathbb{Z} \times \overline{\mathbb{Q}}$ which is given as the union of the elements $(0,q)$ $(1 \leq q \leq \infty)$ and $(1,q)$ $(-\infty < q \leq 0)$. For convenience we will write q rather than (n,q) for those elements; in particular we write 1 for $(0,1)$ and 0 for $(1,0)$. We denote the ordering by $\leq$.

Theorem 4. *Let Λ_1 be the tubular canonical $\mathbb{R}$-algebra which is given by the species*

$$\begin{tikzcd} & \mathbb{C} \arrow[dr, "\mathbb{H}"] & \\ \mathbb{R} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{H}"] \arrow[dr, "\mathbb{C}'"'] & & \mathbb{H} \\ & \mathbb{C}' \arrow[ur, "\mathbb{H}"'] & \end{tikzcd}$$

with ideal of relations generated by

$$u_1 \otimes u_1 - u_0, \; u_2 \otimes u_2 - u_0,$$

where u_1 (resp. u_2, u_0) denotes the element in the tensor algebra in the "upper arm" (resp. "lower arm", "middle" arrow) associated with the identity $\mathbf{1} \in \mathbb{H}$. Then the Auslander-Reiten quiver of Λ_1 consists of the following:

1. *the preprojective component and the preinjective component.*
2. *For each $q \in [1,\infty] \vee]{-\infty},0]$ a separating tubular family $\mathcal{S}_q$ admitting only two non-homogeneous tubes, where for each $q \neq 0$, 1 the family $\mathcal{S}_q$ is stable, $\mathcal{S}_1$ contains a projective indecomposable, $\mathcal{S}_0$ contains an injective indecomposable. For q, $q' \in [1,\infty] \vee]{-\infty},0]$ we have $\mathrm{Hom}(\mathcal{S}_q, \mathcal{S}_{q'}) \neq 0$ if and only if $q \leq q'$. Moreover, if $q = \frac{a}{b}$, where a and b are coprime, then each simple object in $\mathcal{S}_q$ has endomorphism ring isomorphic to $\mathbb{C}$ if b is even, whereas each of the skew-fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ will occur as the endomorphism ring of a simple object in $\mathcal{S}_q$ if b is odd.*

Concerning generic modules we prove

Theorem 5. *Let Λ_1 be the tubular canonical $\mathbb{R}$-algebra from Theorem 4. Then there is a bijection $q \mapsto G_q$ between the set of elements $q \in [1,\infty] \vee]{-\infty},0]$ and the set of (isoclasses of) generic Λ_1-modules; moreover, for all generic modules G_q, $G_{q'}$ the relation $\mathrm{Hom}(G_q, G_{q'}) \neq 0$ holds if and only if $q \leq q'$ and with $q = \frac{a}{b}$, where a and b are coprime, we have that $\mathrm{End}(G_q)$ is isomorphic to the quotient field of $\mathbb{R}[X,Y]/(X^2+Y^2+1)$ in case b is even, whereas $\mathrm{End}(G_q)$ is isomorphic to the skew function field $\mathbb{C}(T, \overline{\cdot})$ (where $\alpha T = T\overline{\alpha}$ for each $\alpha \in \mathbb{C}$) in case b is odd.*

3. A graded factorial algebra over the reals

Let R be the affine $\mathbb{R}$-algebra $\mathbb{R}[X,Y,Z]/(X^2+Y^4+Z^4) = \mathbb{R}[x,y,z]$ which is graded by the abelian group H generated by $\vec{x} = \deg x$, $\vec{y} = \deg y$, $\vec{z} = \deg z$ with relations $\vec{x} = 2\vec{y} = 2\vec{z}$. Note that H has a torsion part which is generated by $\vec{y} - \vec{z}$. Graded factoriality of R follows by using a slightly more general method as in [20, Thm. 4.1] from the graded factoriality of the $\mathbb{Z}$-graded algebra $\mathbb{R}[X,Y,Z]/(X^2 + Y^2 + Z^2)$.

Let $\mathbb{X}_1$ be the graded projective spectrum of R (compare [7]). We consider the category $\mathcal{H} = \mathrm{coh}(\mathbb{X}_1) = \frac{\mathrm{mod}^H(R)}{\mathrm{mod}_0^H(R)}$ of H-graded coherent sheaves over $\mathbb{X}_1$ (as in [7]). For each $h \in H$ and $\mathcal{F} \in \mathcal{H}$ the coherent sheaf which is obtained from $\mathcal{F}$ by degree-shift by h is denoted by $\mathcal{F}(h)$. Denote by $\mathrm{K}_0(\mathbb{X}_1)$ the Grothendieck group of $\mathrm{coh}(\mathbb{X}_1)$, which is equipped with the Euler form $\langle -,- \rangle$ defined by

$$\langle [\mathcal{F}], [\mathcal{G}] \rangle = \dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{X}_1}(\mathcal{F}, \mathcal{G}) - \dim_{\mathbb{R}} \mathrm{Ext}^1_{\mathbb{X}_1}(\mathcal{F}, \mathcal{G}).$$

Note that the category $\mathcal{H}$ is hereditary. Let $\omega = \vec{x} - \vec{y} - \vec{z}$. Then we have (as in [6, Thm. 2.2]) Serre duality in the form

$$\mathrm{D}\,\mathrm{Ext}^1_{\mathbb{X}_1}(\mathcal{F},\mathcal{G}) \simeq \mathrm{Hom}_{\mathbb{X}_1}(\mathcal{G},\mathcal{F}(\omega))$$

for $\mathcal{F}$, $\mathcal{G} \in \mathrm{coh}(\mathbb{X}_1)$, where D denotes the duality $\mathrm{Hom}_{\mathbb{R}}(-,\mathbb{R})$. Moreover, as in [6, Cor. 2.3] one shows that degree-shift by ω coincides with the Auslander-Reiten translation $\tau : \mathcal{H} \longrightarrow \mathcal{H}$, which is an equivalence of categories: $\tau\mathcal{F} \simeq \mathcal{F}(\omega)$. The Auslander-Reiten translation induces an automorphism on $\mathrm{K}_0(\mathbb{X}_1)$, called *Coxeter transformation*, which preserves the Euler form. The *radical* $\mathrm{Rad}\,\mathrm{K}_0(\mathbb{X}_1)$ consists of the fixed points of the Coxeter transformation.

We denote by $\mathcal{O}$ (and often also by L) the structure sheaf over $\mathbb{X}_1$, and by $\mathcal{S}_x$ ($\mathcal{S}_y$, $\mathcal{S}_z$) the simple sheaf which is concentrated in the closed point Rx (Ry, Rz, resp.) such that $\mathrm{Ext}^1_{\mathbb{X}_1}(\mathcal{S}_x,\mathcal{O}) \neq 0$ (resp., the analogous conditions for $\mathcal{S}_y$, $\mathcal{S}_z$); the dimension over $\mathbb{R}$ of these non-zero spaces is two. The sheaves $\mathcal{S}_y$, $\mathcal{S}_z$ are exceptional (lying in tubes of rank two), whereas $\mathcal{S}_x$ is ordinary (that is, lying in a homogeneous tube). For each $\mathcal{F} \in \mathrm{coh}(\mathbb{X}_1)$ we have $\mathrm{rk}\,\mathcal{F} = \frac{1}{2}\dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{X}_1}(\mathcal{F},\mathcal{S}_x)$ and $\deg \mathcal{F} = \frac{1}{2}\dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{X}_1}(\mathcal{O},\mathcal{F}) + \frac{1}{2}\dim_{\mathbb{R}} \mathrm{Hom}_{\mathbb{X}_1}(\mathcal{O},\mathcal{F}(\omega))$.

Let $0 \longrightarrow \mathcal{O}^2 \longrightarrow L_i \longrightarrow \mathcal{S}_i \longrightarrow 0$ be the $\mathcal{O}$-co-universal extensions of $\mathcal{S}_1 := \mathcal{S}_y$ and $\mathcal{S}_2 := \mathcal{S}_z$, resp., and let $0 \longrightarrow \mathcal{O}^2 \longrightarrow \overline{L} \longrightarrow \mathcal{S}_x \longrightarrow 0$ be the $\mathcal{O}$-co-universal extension of $\mathcal{S}_x$. (For the notion of (co-) universal extensions see [16, Ch. 3].) Then the full subcategory

$$\begin{tikzcd} & L_1 \arrow[dr] & \\ \mathcal{O} \arrow[ur] \arrow[rr] \arrow[dr] & & \overline{L} \\ & L_2 \arrow[ur] & \end{tikzcd}$$

forms a tilting bundle T_1 in $\mathcal{H}$. By definition the canonical algebra Λ_1 from Theorem 4 is the endomorphism algebra of T_1 (compare [16, Prop. 5.5]). Furthermore, it follows that $\mathbb{X}_1$ is an exceptional curve (compare [15]). Since $2 \cdot \omega = 0$ (thus, the Coxeter transformation has order 2), this exceptional curve is tubular (compare [13, Thm. 8.3]), and hence Λ_1 is also tubular.

Moreover, if $\mathrm{mod}_0(\Lambda_1)$ denotes the "central" separating tubular family over Λ_1 (which consists of the Λ_1-modules of rank 0 (=defect 0, [22])), then it is easy to see that $\mathcal{H}(\Lambda_1, \mathrm{mod}_0(\Lambda_1)) = \mathrm{coh}(\mathbb{X}_1)$.

Lemma 6. *Let $\mathcal{S}$ be a simple sheaf over $\mathbb{X}_1$. Then the endomorphism ring of $\mathcal{S}$ is isomorphic to the field of complex numbers $\mathbb{C}$.*

Proof. By reduction of weights (see [12, Lemma 10], also [7, Thm. 9.5]) it is sufficient to show that the endomorphism ring of each simple sheaf over the projective spectrum of the $\mathbb{Z}$-graded algebra $\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2)$ is isomorphic to $\mathbb{C}$. $\square$

4. The slope categories

Let $\mathbb{X}_1$ be the tubular exceptional curve as in the preceding section, and let $\mathbb{X}$ be an arbitrary tubular exceptional curve. Since each indecomposable coherent sheaf over $\mathbb{X}$ is semi-stable, we have

$$\operatorname{coh}(\mathbb{X}) = \bigvee_{q\in\overline{\mathbb{Q}}} \mathcal{H}^{(q)},$$

where the symbol $\bigvee$ indicates that $\mathcal{H} = \operatorname{coh}(\mathbb{X})$ is the additive closure of the union of the slope categories $\mathcal{H}^{(q)}$ ($q \in \overline{\mathbb{Q}}$), and additionally (under the assumption $\mathcal{H}^{(q)}$, $\mathcal{H}^{(q')} \neq 0$)

$$\operatorname{Hom}_{\mathbb{X}}(\mathcal{H}^{(q)}, \mathcal{H}^{(q')}) \neq 0 \iff q \leq q'.$$

("$\Longrightarrow$" by semi-stability, "$\Longleftarrow$" by the Riemann-Roch formula [13, Thm. 9.2].) Since $\operatorname{coh}(\mathbb{X})$ is hereditary, $\mathcal{D} = \mathrm{D}^b(\mathbb{X})$ is the repetitive category of $\operatorname{coh}(\mathbb{X})$, the additive closure of the copies $\mathcal{H}^{(q)}[n]$ ($q \in \overline{\mathbb{Q}}$, $n \in \mathbb{Z}$) (see [8]). The natural automorphism on $\mathcal{D}$ sending $X \in \mathcal{H}[n]$ to the copy $X[1] \in \mathcal{H}[n+1]$ is called *translation functor*.

For each $q \in \overline{\mathbb{Q}}$ let $\mathcal{H}\langle q\rangle$ be the subcategory in $\mathcal{D}$ defined by

$$\mathcal{H}\langle q\rangle = \mathcal{H}_{-}^{(q)}[-1] \vee \mathcal{H}_{+}^{(q)} \vee \mathcal{H}^{(q)},$$

where

$$\mathcal{H}_{+}^{(q)} = \bigvee_{-\infty<r<q} \mathcal{H}^{(r)}, \quad \mathcal{H}_{-}^{(q)} = \bigvee_{q<r\leq\infty} \mathcal{H}^{(r)}.$$

Moreover

$$\mathcal{D} = \mathcal{D}_{+}^{(q)} \vee \mathcal{H}^{(q)} \vee \mathcal{D}_{-}^{(q)},$$

where

$$\mathcal{D}_{+}^{(q)} = \bigvee_{n<0} \mathcal{H}[n] \vee \mathcal{H}_{+}^{(q)} \quad \text{and} \quad \mathcal{D}_{-}^{(q)} = \mathcal{H}_{-}^{(q)} \vee \bigvee_{n>0} \mathcal{H}[n].$$

Proposition 7 (Lenzing). *Let $\mathbb{X}$ be a tubular exceptional curve. Let $q \in \overline{\mathbb{Q}}$ and assume that $\mathcal{H}^{(q)}$ is non-trivial. Then the category $\mathcal{H}\langle q\rangle$ is of the form $\operatorname{coh}(\mathbb{X}\langle q\rangle)$, where $\mathbb{X}\langle q\rangle$ is a tubular exceptional curve.*

Proof. It is enough to show ([15]) that $\mathcal{H}\langle q\rangle$ is an abelian, hereditary category which is noetherian with no non-zero projectives and admits a tilting object. Since the pair $(\mathcal{H}_{+}\langle q\rangle \vee \mathcal{H}^{(q)}, \mathcal{H}_{-}\langle q\rangle)$ defines a cut in $\mathcal{H}$ (in the terminology of [18]) it follows from [9] that $\mathcal{H}\langle q\rangle$ is abelian. It is sufficient to show that $\mathcal{H}\langle q\rangle$ is noetherian, the remaining assertions then follow easily. Let S be a simple object in $\mathcal{H}^{(q)}$, lying in a tube of rank p. The element $\mathbf{w}_q = \sum_{j=0}^{p-1}[\tau^j S]$ in the Grothendieck group defines a (non-normalized) rank function for objects of $\mathcal{H}\langle q\rangle$ by means of $\operatorname{rk}_q X = \langle [X], \mathbf{w}_q\rangle$. For all $X \in \mathcal{H}\langle q\rangle$ we have $\operatorname{rk}_q X \geq 0$, and $\operatorname{rk}_q X = 0$ if and only if $X \in \mathcal{H}^{(q)}$. Now it follows as in [16, Thm. 6.1] that $\mathcal{H}\langle q\rangle$ is noetherian. $\square$

Proposition 8 (Lenzing). *Let $\mathbb{X}$ be a tubular exceptional curve. Let $q \in \overline{\mathbb{Q}}$ and assume that $\mathcal{H}^{(q)}$ is non-trivial. The slope category $\mathcal{H}^{(q)}$ is a separating tubular family in $\mathrm{D}^b(\mathbb{X})$.*

Proof. By means of the preceding proposition, switching the roles of $\mathcal{H}$ and $\mathcal{H}\langle q\rangle$ we may assume that $q = \infty$. Let X and Y be indecomposable in $\mathcal{D}$, $X \in \mathcal{D}_{+}^{(\infty)}$, $Y \in \mathcal{D}_{-}^{(\infty)}$, and let $f : X \longrightarrow Y$ be a non-zero morphism. Heredity of $\mathcal{H}$ implies $X \in \mathcal{H}$ and $Y \in \mathcal{H}[1]$, hence $Y = Z[1]$ with $Z \in \mathcal{H}$. Let a be in $\mathbb{X}$ and denote by

σ_a the shift functor corresponding to a (compare [16, S10]), and let $X_n = \sigma_a^{-n} X$. For $n \gg 0$ we get $\mu X_n < \mu Z$. Let $X' = X_n$. The canonical exact sequences

$$0 \longrightarrow X_k \longrightarrow X_{k-1} \longrightarrow V_k \longrightarrow 0$$

with V_k semisimple in $\mathcal{U}_a$ yield an exact sequence

$$0 \longrightarrow X' \longrightarrow X \xrightarrow{u} V \longrightarrow 0$$

with $V \in \mathcal{U}_a$. Application of $\mathrm{Hom}_{\mathcal{D}}(-, Y)$ leads to the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(V, Y) \xrightarrow{-\circ u} \mathrm{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}(X', Z) \longrightarrow 0,$$

where $\mathrm{Ext}^1_{\mathcal{H}}(X', Z) \simeq \mathrm{D}\,\mathrm{Hom}_{\mathcal{H}}(Z, \tau X') = 0$, since $\mu Z > \mu X' = \mu \tau X'$. Hence each morphism $f : X \longrightarrow Y$ factors through $u : X \longrightarrow V$. $\square$

Corollary 9. *$\mathcal{H}^{(0)}$ and $\mathcal{H}^{(\infty)}$ are separating tubular families in $\mathrm{D}^b(\mathbb{X}_1)$.*

Proof. Since $\mathcal{O} \in \mathcal{H}^{(0)}$ we have $\mathcal{H}^{(0)} \neq 0$. $\square$

The shift functors on $\mathrm{D}^b(\mathbb{X})$ associated to tubes in *any* separating tubular family [16, S10] are also called *tubular mutations* [19].

Proposition 10. *Denote by σ_L and σ_1 the tubular mutations with respect to the tubes which are generated by the Auslander-Reiten orbits of $L = \mathcal{O}$ and $\mathcal{S}_1$, respectively. Let U be the subgroup of $\mathrm{Aut}(\mathrm{D}^b(\mathbb{X}_1))$ which is generated by σ_L, σ_1 and the translation functor. Then U acts on the set of slope categories $\mathcal{H}^{(q)}[n]$ ($q \in \overline{\mathbb{Q}}$, $n \in \mathbb{Z}$) with exactly two orbits. Moreover, each slope category $\mathcal{H}^{(q)}$ is non-trivial.*

Proof. σ_L and σ_1 induce automorphisms $\overline{\sigma}_L$ and $\overline{\sigma}_1$ on the Grothendieck group (see [16, S10]): For each $\mathbf{x} \in \mathrm{K}_0(\mathbb{X}_1)$ we have

$$\overline{\sigma}_L(\mathbf{x}) = \mathbf{x} - \langle [L], \mathbf{x} \rangle [L] - \langle [\tau L], \mathbf{x} \rangle [\tau L]$$

and

$$\overline{\sigma}_1(\mathbf{x}) = \mathbf{x} - \frac{1}{2}\langle [\mathcal{S}_1], \mathbf{x} \rangle [\mathcal{S}_1] - \frac{1}{2}\langle [\tau \mathcal{S}_1], \mathbf{x} \rangle [\tau \mathcal{S}_1];$$

moreover, $\mu(\overline{\sigma}_L \mathbf{x}) = \frac{\mu(\mathbf{x})}{1 - 2\mu(\mathbf{x})}$ and $\mu(\overline{\sigma}_1 \mathbf{x}) = \mu(\mathbf{x}) + 1$. As is easily shown, the group generated by the maps $q \mapsto \frac{q}{1-2q}$ and $q \mapsto q + 1$ acts on $\overline{\mathbb{Q}}$ with two orbits which are given by the set of fractions where the denominator is even or odd, respectively. $\square$

Corollary 11. *Let σ be an element of $\mathrm{Aut}(\mathrm{D}^b(\mathbb{X}_1))$. Then $\sigma(\mathcal{H}^{(q)}[n])$ is of the form $\mathcal{H}^{(q')}[n']$ for some $q' \in \overline{\mathbb{Q}}$, $n' \in \mathbb{Z}$.*

Proof. Let X, $Y \in \mathcal{H}^{(q)}[n]$ be indecomposable. Since σ induces an automorphism of $\mathrm{Rad}\,\mathrm{K}_0(\mathbb{X}_1)$, and hence (by a result analogous to [17, 2.6]) a bijective map $\overline{\sigma} : \overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}$, it follows that $\sigma X \in \mathcal{H}^{(\overline{\sigma} q)}[n']$ and $\sigma Y \in \mathcal{H}^{(\overline{\sigma} q)}[n'']$ for some n', $n'' \in \mathbb{Z}$. Since for $q' < \overline{\sigma} q$ ($q' < q$, resp.) there are non-zero morphisms from any non-zero element in $\mathcal{H}^{(q')}$ into each tube in $\mathcal{H}^{(\overline{\sigma} q)}$ ($\mathcal{H}^{(q)}$, resp.), we get $n' = n''$. $\square$

Corollary 12. *The group $\mathrm{Aut}\,\mathrm{D}^b(\mathbb{X}_1)$ acts with exactly two orbits on the set of slope categories $\mathcal{H}^{(q)}[n]$ ($q \in \overline{\mathbb{Q}}$, $n \in \mathbb{Z}$). Moreover, slope categories belonging to different orbits are non-equivalent.*

Proof. Each endomorphism ring of a simple object in $\mathcal{H}^{(\infty)}$ is isomorphic to $\mathbb{C}$ (by Lemma 6), and $\mathcal{O} \in \mathcal{H}^{(0)}$ is a simple object and has an endomorphism ring which is isomorphic to the degree zero component of the graded algebra R, which is the field $\mathbb{R}$. Therefore the categories $\mathcal{H}^{(0)}$ and $\mathcal{H}^{(\infty)}$ are not equivalent. $\square$

This proves Theorem 3. It follows easily from Cor. 11 that $\operatorname{Aut} \mathrm{D}^b(\mathbb{X}_1)$ acts also on the set of all $\mathcal{H}\langle q\rangle[n]$ ($q \in \overline{\mathbb{Q}}$, $n \in \mathbb{Z}$), with the same number of orbits.

5. A second tilting bundle

Let $\mathbb{X}_1$ be the tubular exceptional curve as in the previous sections. We call a linear map $r : \mathrm{K}_0(\mathbb{X}_1) \longrightarrow \mathbb{Z}$ a *rank function*, if it is *normalized* (that is, surjective) and compatible with the Coxeter transformation on $\mathrm{K}_0(\mathbb{X}_1)$. For example, the rank of sheaves in $\operatorname{coh}(\mathbb{X}_1)$ (as defined in Section 2) induces a rank function.

Let $\mathbb{X}_2$ be the tubular exceptional curve such that $\mathcal{H}\langle 0\rangle = \operatorname{coh}(\mathbb{X}_2)$ (by Prop. 7). Then $\mathcal{H}\langle 0\rangle$ admits a torsion-free tilting object ([16]) constructed as follows: The separating tubular family $\mathcal{H}^{(0)}$ induces a rank function $\mathrm{rk}_0 : \mathrm{K}_0(\mathbb{X}_1) \longrightarrow \mathbb{Z}$ (as constructed in the proof of Proposition 7, but additionally normalized), which is uniquely determined by the property that each object in $\mathcal{H}^{(0)}$ has rk_0 zero, and the torsion-free objects in $\mathcal{H}\langle 0\rangle$ have positive rk_0. There are two exceptional tubes $T_1^{(0)}$ and $T_2^{(0)}$ in $\mathcal{H}^{(0)}$, each of rank two. The structure sheaf $\mathcal{O}$ is lying in one of them. By [16] there is an exceptional $L_0 \in \mathcal{H}\langle 0\rangle$ such that $\mathrm{rk}_0 L_0 = 1$, and $\operatorname{Ext}^1(\mathcal{S}_i^{(0)}, L_0) \neq 0$ for exactly one simple exceptional object $\mathcal{S}_i^{(0)}$ in the exceptional tube $T_i^{(0)}$ in $\mathcal{H}^{(0)}$ ($i = 1, 2$). Moreover, let $\mathcal{S}^{(0)}$ be the direct sum of a τ-orbit of a simple object in $\mathcal{H}^{(0)}$ concentrated in a rational point (see [16]). Then let $0 \longrightarrow L_0^{m_i} \longrightarrow L_i^{(0)} \longrightarrow \mathcal{S}_i^{(0)} \longrightarrow 0$ be the L_0-co-universal extensions of $\mathcal{S}_1^{(0)}$ and $\mathcal{S}_2^{(0)}$, respectively, and let $0 \longrightarrow L_0^m \longrightarrow \overline{L}_0 \longrightarrow \mathcal{S}^{(0)} \longrightarrow 0$ be the L_0-co-universal extension of $\mathcal{S}^{(0)}$. By [16] we get a tilting bundle T_2, defined by the full subcategory

$$\begin{tikzcd}
& L_1^{(0)} \arrow[dr] & \\
L_0 \arrow[ur] \arrow[rr] \arrow[dr] & & \overline{L}_0 \\
& L_2^{(0)} \arrow[ur] &
\end{tikzcd}$$

whose endomorphism algebra is a canonical algebra Λ_2. Here, $M := \operatorname{Hom}(L_0, \overline{L}_0)$ is a tame $\operatorname{End}(\overline{L}_0) - \operatorname{End}(L_0)$-bimodule over $\mathbb{R}$, which admits a simple regular representation with endomorphism ring isomorphic to $\operatorname{End}(\mathcal{O}) = \mathbb{R}$. By the classification of tame bimodules over $\mathbb{R}$ ([5]) then M must be one of the bimodules ${}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}} \oplus {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}$ or ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\overline{\mathbb{C}}}$, where $\mathbb{C}$ is acting on the second component from the right via conjugation. But the first case is impossible, since it does not lead to a tubular symbol (see [13] and [16, Ch. 5]). Using the fact, that the Grothendieck groups of Λ_1 and Λ_2 coincide, and studying the list of tubular symbols [13], it follows that the underlying species of Λ_2 is of the form

$$\begin{tikzcd}
& \mathbb{R} \arrow[dr, "\mathbb{C}"] & \\
\mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}\oplus\overline{\mathbb{C}}"] \arrow[dr, "\mathbb{H}"'] & & \mathbb{C} \\
& \mathbb{H} \arrow[ur, "\mathbb{H}"'] &
\end{tikzcd}$$

The "central" separating tubular family $\mathrm{mod}_0(\Lambda_2)$ in $\mathrm{mod}(\Lambda_2)$ is given by $\mathcal{H}^{(0)}$, and we have $\mathrm{coh}(\mathbb{X}_2) = \mathcal{H}(\Lambda_2, \mathrm{mod}_0(\Lambda_2))$.

We have $\mathrm{D}^b(\mathbb{X}_1) = \mathrm{D}^b(\mathbb{X}_2)$, and shifting T_2 in the derived category shows that Λ_2 can also be realized as endomorphism algebra of a tilting bundle in $\mathrm{coh}(\mathbb{X}_1)$. Moreover, Λ_1 and Λ_2 are tilting-equivalent, in particular, derived-equivalent.

This proves Theorem 1.

Lemma 13. *The function field of $\mathbb{X}_1$ is isomorphic to the quotient field of the ring $\mathbb{R}[X,Y]/(X^2+Y^2+1)$, the function field of $\mathbb{X}_2$ is isomorphic to the skew rational function field $\mathbb{C}(T,\overline{\cdot})$, where $\alpha T = T\overline{\alpha}$ for each $\alpha \in \mathbb{C}$.*

Proof. By reduction of weights we can assume that we are in the underlying homogeneous situation (that is, all tubes are of rank one). Using the methods of [2] and [6], we see that the function field is isomorphic to the degree zero component of the total ring of homogeneous quotients of the preprojective algebra associated to the Auslander-Reiten orbit of a preprojective rank one module over the tame hereditary bimodule algebra $\begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{pmatrix}$ in the case of $\mathbb{X}_1$, and $\begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C}\oplus\overline{\mathbb{C}} & \mathbb{C} \end{pmatrix}$ in the case $\mathbb{X}_2$; in the first case, the preprojective algebra is $\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2)$, graded by total degree (compare [11]); in the second case it is isomorphic to $\mathbb{C}[X,Y,\overline{\cdot}]^{(2)}$, where X is central and $\alpha Y = Y\overline{\alpha}$ for all $\alpha \in \mathbb{C}$, and where the index (2) means, that we only consider the even degree part (this follows by similar arguments as in [11]). ☐

This proves Theorem 2. Moreover, with the same arguments as in [17, Thm. 5.7], we can reconstruct $\mathrm{mod}(\Lambda_1)$ from $\mathrm{coh}(\mathbb{X}_1)$, which proves Theorem 4.

6. Generic modules

We keep the notation from the previous sections. In particular, let $\Lambda = \Lambda_1$ be the tubular canonical algebra from Theorem 4, let $\mathbb{X} = \mathbb{X}_1$ be the curve associated to $\mathbb{R}[X,Y,Z]/(X^2+Y^4+Z^4)$ and T be the tilting bundle T_1 such that $\mathrm{End}(T) = \Lambda$. The aim of this section is to prove Theorem 5. The arguments are similar to the arguments in [14]. Therefore we will be very brief here.

First recall that an indecomposable object G of $\mathrm{D}^b(\mathrm{Mod}(\Lambda)) = \mathrm{D}^b(\mathrm{Qcoh}(\mathbb{X}))$ is called *generic* [14], if it does not belong to $\mathrm{D}^b(\mathrm{mod}(\Lambda)) = \mathrm{D}^b(\mathrm{coh}(\mathbb{X}))$, and $\mathrm{Hom}(E,G)$ has finite $\mathrm{End}(G)$-length for each E from $\mathrm{D}^b(\mathrm{mod}(\Lambda)) = \mathrm{D}^b(\mathrm{coh}(\mathbb{X}))$.

A Λ-module G is generic if and only if G is generic as an object of $\mathrm{D}^b(\mathrm{Mod}(\Lambda))$. An indecomposable quasi-coherent sheaf G is generic in $\mathrm{D}^b(\mathrm{Qcoh}(\mathbb{X}))$ if and only if G is not coherent and both $\mathrm{Hom}(T,G)$ and $\mathrm{Ext}^1(T,G)$ have finite $\mathrm{End}(G)$-length; those sheaves are called generic (compare [14]).

Let G be a generic module (resp. sheaf). Then G defines a linear form on $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\mathbb{X})$ given on classes $[E]$ of finite dimensional Λ-modules (resp. of coherent sheaves) by the expression

$$\langle [E], G] = \sum_{n=0}^{\infty} (-1)^n \lg_{\mathrm{End}(G)}(\mathrm{Ext}^n(E,G)).$$

Since the Euler form of Λ is non-degenerate there exists a unique element $[G] \in \mathrm{K}_0(\Lambda)$ such that

$$\langle [E], G] = \frac{1}{c_G} \langle [E], [G] \rangle$$

for all $E \in \mathrm{mod}(\Lambda)$ (or, equivalently, for all $E \in \mathrm{coh}(\mathbb{X})$) for some factor $c_G \in \mathbb{N}$ and such that $[G]$ is not divisible by some natural number $m \geq 2$. The element $[G]$ is called *characteristic class* of G ([14]).

Let $\mathbf{u} = [\mathcal{O}] + [\tau\mathcal{O}]$ and $\mathbf{w} = [\mathcal{S}_x]$, where $\mathcal{S}_x$ is the simple sheaf concentrated in the ordinary point $x \in \mathbb{X}$ associated to the variable X (compare Section 3). Then $\mathbf{u}$, $\mathbf{w}$ form a $\mathbb{Z}$-basis of $\mathrm{Rad}\,\mathrm{K}_0(\mathbb{X})$. Recall that for each $\mathcal{F} \in \mathrm{coh}(\mathbb{X})$ the rank $\mathrm{rk}\,\mathcal{F}$ coincides with $\frac{1}{2}\langle [\mathcal{F}], \mathbf{w} \rangle$ and the degree $\deg \mathcal{F}$ with $\frac{1}{2}\langle \mathbf{u}, [\mathcal{F}] \rangle$. Let G be a generic module. Define $\mathrm{rk}\, G := \frac{1}{2}\langle [G], \mathbf{w} \rangle$, $\deg G := \frac{1}{2}\langle \mathbf{u}, [G] \rangle$ and the slope $\mu G := \deg G / \mathrm{rk}\, G$.

Lemma 14. *Let $\mathcal{K}$ be an injective hull of $L = \mathcal{O}$ in* $\mathrm{Qcoh}(\mathbb{X})$. *Then*

(1) *$\mathcal{K}$ is torsion-free.*

(2) *$\mathcal{K}$ is preserved by all shifts* (*as defined in* [16, S10]). *Namely, $\mathcal{K}(a) \simeq \mathcal{K}$ for each $a \in \mathbb{X}$.*

(3) *If L' is a line bundle* (*that is, a sheaf of rank one*), *then $\mathcal{K}$ is the injective hull of L'.*

(4) $\mathrm{End}(\mathcal{K})$ *is isomorphic to the function field of $\mathbb{X}$, that is, isomorphic to the quotient field of* $\mathbb{R}[X,Y]/(X^2+Y^2+1)$.

Proof. (1) Assume that there is a simple subsheaf S of $\mathcal{K}$. Since $\mathcal{K}$ is the injective hull of L then S is also a subsheaf of L, which is impossible, since L is torsion-free.

(2) Let $a \in \mathbb{X}$ and let $\mathcal{S}_a$ be a simple sheaf concentrated in a and lying in a tube of rank $p(a)$. Then $\mathcal{K}(a)$ is defined by the universal exact sequence (compare the definition of the shift functor σ_a in [16, S10])

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{K}(a) \longrightarrow \textstyle\bigoplus_{j=1}^{p(a)} \mathrm{Ext}^1(\tau^j \mathcal{S}_a, \mathcal{K}) \otimes_{\mathrm{End}(\mathcal{S}_a)} \tau^j \mathcal{S}_a \longrightarrow 0,$$

where the last term is zero since $\mathcal{K}$ is injective.

(3) For $n \gg 0$ we have $\mathrm{Hom}(L', L(na)) \neq 0$ (by [16, S15]; here, $L(na) := \sigma_a^n(L)$), thus L' embeds into $\mathcal{K}$, therefore the injective hull is a direct summand of $\mathcal{K}$. Since by (4) $\mathcal{K}$ is indecomposable, we get our assertion.

(4) Denote by $\widetilde{L}$ and $\widetilde{\mathcal{K}}$ the corresponding elements in the quotient category $\frac{\mathrm{Qcoh}(\mathbb{X})}{\mathrm{Qcoh}_0(\mathbb{X})}$, modulo locally finite objects (which are by definition the filtered unions of objects of finite length). It is easy to see that $\mathrm{End}(\mathcal{K}) \simeq \mathrm{End}(\widetilde{\mathcal{K}}) \simeq \mathrm{End}(\widetilde{L})$. Moreover, by the arguments of [2, Cor. 4.7] the category $\frac{\mathrm{Qcoh}(\mathbb{X})}{\mathrm{Qcoh}_0(\mathbb{X})}$ is semisimple with $\widetilde{L}$ as its unique simple object, therefore $\frac{\mathrm{Qcoh}(\mathbb{X})}{\mathrm{Qcoh}_0(\mathbb{X})} \simeq \mathrm{Mod}(D)$, where $D \simeq \mathrm{End}(\widetilde{L})$ is the function field. $\square$

Corollary 15. *$\mathcal{K}$ is a generic sheaf with characteristic class $[\mathcal{K}] = \mathbf{w}$ and rank zero.*

Proof. Invoking injectivity of $\mathcal{K}$ we have for each $E \in \operatorname{coh}(\mathbb{X})$

$$\langle [E], G] = \dim_{\operatorname{End}(\mathcal{K})} \operatorname{Hom}(E, \mathcal{K}) = \operatorname{rk} E = \frac{1}{2}\langle [E], \mathbf{w}\rangle.$$

(For E a sheaf of finite length or a line bundle this follows by the preceding lemma; for arbitrary torsion-free sheaf E use a line bundle filtration as in [6, 2.6].) □

Proposition 16. *Each generic sheaf of rank zero is isomorphic to $\mathcal{K}$.*

Proof. As in [14, Prop. 4.2]. □

Corollary 17. *Each generic sheaf G has a well-defined slope $\mu G \in \overline{\mathbb{Q}}$.*

Proof. Follows from $\deg \mathcal{K} = \frac{1}{2}\langle \mathbf{u}, \mathbf{w}\rangle = 2$. □

Proposition 18. (1) *For each $q \in \overline{\mathbb{Q}}$ there is (up to isomorphism) exactly one generic sheaf G_q of slope q.*

(2) *For each q, $q' \in \overline{\mathbb{Q}}$ we have $\operatorname{Hom}(G_q, G_{q'}) \neq 0$ if and only if $q \leq q'$.*

(3) *For each $q \in \overline{\mathbb{Q}}$ we have $\operatorname{Ext}^1(G_q, G_q) = 0$ and $\operatorname{End}(G_q)$ is isomorphic to the function field of the tubular exceptional curve $\mathbb{X}\langle q\rangle$.*

Proof. Let $q \in \overline{\mathbb{Q}}$. Let $\mathbb{X}\langle q\rangle$ be the tubular exceptional curve such that $\operatorname{coh}(\mathbb{X}\langle q\rangle) = \mathcal{H}\langle q\rangle$ and $\operatorname{coh}_0(\mathbb{X}\langle q\rangle) = \mathcal{H}^{(q)}$. The results above also hold for $\mathbb{X}\langle q\rangle$ instead of $\mathbb{X}$. Statement (2) follows with the same arguments as in [14, Prop. 4.9]. □

Proposition 19. *A Λ-module G is generic if and only if it is isomorphic to either*
(1) *G_q where $1 \leq q \leq \infty$, or*
(2) *$G_q[1]$ where $q \leq 0$.*

Proof. This follows as in [14, Thm. 5.1]. □

The results of this section together with Lemma 13 prove Theorem 5.

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