One-parameter families
for finite dimensional algebras

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Abstract

In these notes we investigate the one-parameter families of indecomposable representations of a finite dimensional algebra over an arbitrary base field. For a possible extension of Drozd’s Tame and Wild Theorem the one-parameter families which arise over tame bimodule algebras play a fundamental role. As a main result we show that such a one-parameter family $X$ is always a (noncommutative) projective curve admitting a projective coordinate algebra which is a (not necessarily commutative) graded factorial domain $R$ (a graded version of a notion going back to Chatters and Jordan). More precisely, there is a natural bijection between the points of $X$ and the homogeneous prime ideals of height one in $R$, and these prime ideals are principal.

As an application we obtain new insights into important invariants including the endomorphism ring of the corresponding generic module (the “function field”), the multiplicities (defined by Ringel in 1979), the automorphism group, and the local data of the points.

The results are also applied to the weighted cases, that is, to the exceptional curves, defined by Lenzing as a generalization of the concept of weighted projective lines to arbitrary base fields. These are just the parametrizing curves of the separating tubular families over canonical algebras in the sense of Ringel and Crawley-Boevey.

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Introduction

In these notes we investigate the structure of the one-parameter families of finite dimensional indecomposable modules over finite dimensional algebras. More precisely, we study the class of one-parameter families which arise over tame bimodules and which is considered to be the natural extension of the class of rational one-parameter families to arbitrary base fields. The geometric structure of these families is completely understood only over algebraically closed fields where they play the key role in Drozd’s Tame and Wild Theorem. In contrast the situation over arbitrary base fields is much more difficult and so far there is no extension of Drozd’s theorem known. One should expect that for such a generalization the one-parameter families arising over tame bimodules are the relevant ones. We will prove a structure theorem for these one-parameter families and get new insights into important invariants including the function field (which is the endomorphism ring of the corresponding generic module), the multiplicities which were introduced by C. M. Ringel in [88], the automorphism group, and the local data of a point.

We use this opportunity also to point out the effects which occur in the representation theory of finite dimensional algebras if an arbitrary base field is allowed. It is sometimes assumed that most theorems remain essentially true over arbitrary fields and that algebraically closed fields are considered only for technical reasons. While this is true of a large part of representation theory, this work shows that it is often not predictable whether new effects occur or not.

The interest in the general situation stems from the fact that in many applications one has to deal with algebras which are defined over fields which are not algebraically closed. For example, the base field of real numbers is of interest for applications in analysis, the field of rational numbers for number theory, finite fields for the applications to quantum groups (like Ringel’s Hall algebra approach), etc. The question on the structure of the one-parameter families is then one of the most basic ones.

The Auslander-Reiten school considers Artin algebras (which includes in particular the class of finite dimensional algebras over fields), and their results are mostly independent of the base ring. By contrast, many results
of modern representation theory are first proved over an algebraically closed field and later generalized. These generalizations often require new proofs, frequently leading to better insights and streamlined arguments even for the algebraically closed case. Some examples for this in the context of hereditary representation theory can be seen in the results of Happel and Reiten about the characterization of hereditary abelian categories with tilting object ([37] and [38]) and in the proof of the transitivity of the braid group action on complete exceptional sequences for hereditary Artin algebras by Ringel ([92], generalizing [19]) and by Meltzer and the author for exceptional curves ([59], summarized in Section 7.1).

We will concentrate here on the hereditary theory, even assuming the existence of a tilting object since this is the suitable framework for studying “rational” one-parameter families for finite dimensional algebras. This part of representation theory is far developed, in some sense explicit, and admits a good approach over arbitrary fields. There are many parallels to the theory of smooth projective curves in algebraic geometry. Moreover, many surprising new effects occur already in this situation and may be considered as model also for a more general setting.

More specifically, we deal with the class of concealed canonical algebras [69] over a field $k$. Such a finite-dimensional algebra $\Sigma$ can be essentially characterized by the fact that its module category admits a stable separating tubular family

$$\operatorname{mod}_0(\Sigma) = \bigoplus_{x \in X} U_x.$$ 

Such a tubular family is parametrized by a set $X$, which is endowed naturally with geometric structure, called an exceptional curve [67]. Our aim is twofold: first, to develop general techniques which allow to describe the geometric structure of these parameter curves more explicitly, and second, to exhibit some new phenomena that occur when an arbitrary base field is allowed.

The categories $U_x$ are connected and uniserial; the indecomposable objects in $U_x$ form a so-called tube. Each $U_x$ contains up to isomorphism only a finite number of simple objects, and for almost all $x$ precisely one simple object. This number is called the weight of $x$, and is also called the rank of the corresponding tube. A tube of rank one is called homogeneous. If all tubes are homogeneous, then we refer to this case as the homogeneous case, otherwise as the weighted case.

An important special case of concealed canonical algebras is the class of tame hereditary algebras. In this context there are a number of inspiring papers. For example, those by Dlab and Ringel on bimodules and hereditary algebras [23, 87, 26, 25, 28] (see additionally [27, 21, 22]), in particular
Ringel’s Rome proceedings paper [88], as well as those by Lenzing [63], Baer, Geigle and Lenzing [6], and by Crawley-Boevey [17], dealing with the structure of the parameter curves for tame hereditary algebras over fields.

Most effects already occur in the homogeneous case, that is, when we have a tame hereditary bimodule algebra \( \Lambda = \left( \begin{array}{cc} G & 0 \\ M & F \end{array} \right) \), where \( M = FM_G \) is a tame bimodule over \( k \). These are the analogues of the Kronecker algebra \( \left( \begin{array}{cc} k & 0 \\ k^2 & k \end{array} \right) \), which is isomorphic to the path algebra of the following quiver.

\[ \bullet \rightarrow \bullet \]

Indeed, by the perpendicular calculus and the insertion of weights many problems for concealed canonical algebras (and exceptional curves) can be reduced to the special class of tame bimodule algebras.

The tame bimodules are fundamental for the study of one-parameter families since one should expect that there is an extension of Drozd’s Tame and Wild Theorem [31] to an arbitrary base field \( k \) in the sense that the finite dimensional indecomposable modules over any non-wild (that is, tame) finite dimensional \( k \)-algebra essentially lie in one-parameter families which are induced by tame bimodules over \( k \).

Over a tame bimodule algebra \( \Lambda \) the category of finite dimensional right \( \Lambda \)-modules has the shape

\[ \text{mod}(\Lambda) = \mathcal{P} \vee \mathcal{R} \vee \mathcal{Q}, \]

that is, \( \text{mod}(\Lambda) \) is the additive closure of \( \mathcal{P} \), \( \mathcal{R} \) and \( \mathcal{Q} \), where \( \mathcal{P} \) is the preprojective component, consisting of the Auslander-Reiten orbits of two indecomposable projective modules, dually \( \mathcal{Q} \) is the preinjective component and \( \mathcal{R} \) consists of the regular modules, whose indecomposable summ ands lie in tubes, that is, \( \mathcal{R} = \text{mod}_0(\Lambda) \) is the separating tubular family with all tubes being homogeneous. Whereas the preprojective and the preinjective component (consisting of exceptional modules) are of a combinatorial nature, and thus their properties are essentially independent of the base field, this is completely different for the regular part.

If \( \overline{k} \) is algebraically closed (which we stress temporarily by using this unusual notation) then the Kronecker algebra \( \Lambda \) is the only tame bimodule \( \overline{k} \)-algebra, and the indecomposable \( \Lambda \)-modules which are regular and of fixed dimension form a one-parameter family, indexed by the projective line \( \mathbb{X} = \mathbb{P}^1(\overline{k}) \), which is \( \overline{k} \cup \{ \infty \} \). In particular, the simple regular \( \Lambda \)-modules are parametrized by \( \mathbb{P}^1(\overline{k}) \). If \( k \) is arbitrary and \( \Lambda \) the Kronecker algebra over \( k \) then the simple regular modules are also parametrized by the projective line \( \mathbb{P}^1(k) \), but now understood in the scheme sense (and not just as \( k \cup \{ \infty \} \)), which corresponds to the fact that in general there are simple regular
modules of higher dimensions. Thus, also for the Kronecker algebra over an arbitrary field there is a good description of the one-parameter family, at least theoretically, since it is in general impossible to describe the points of $\mathbb{P}^1(k)$ explicitly, as one sees in the example $k = \mathbb{Q}$. Since the Kronecker bimodules form only one very special case in the huge class of tame bimodules over fields it is a highly non-trivial task to describe the geometric structure of the index set $X$ of the simple regular modules in general.

Instead of working with the module category $\text{mod}(\Lambda)$, we prefer to use the category $\mathcal{H} = \mathcal{H}(\Lambda)$, which is defined in the derived category $D^b(\text{mod}(\Lambda))$ by

$$\mathcal{H} = \mathcal{Q}[−1] \vee \mathcal{P} \vee \mathcal{R}.$$  

In other words, $\mathcal{H}$ is obtained from $\text{mod}(\Lambda)$ by shifting the preinjective component to the left and gluing it to the preprojective component, creating thus a category without non-zero projectives or injectives.

$\mathcal{H}$ is again a hereditary category, the regular $\Lambda$-modules become the objects of finite length in $\mathcal{H}$, the Auslander-Reiten translation gives rise to an autoequivalence on $\mathcal{H}$, and $\mathcal{H}$ has a tilting object with endomorphism ring $\Lambda$. The objects of $\mathcal{Q}[−1] \vee \mathcal{P}$ are also called vector bundles. Denote by $L$ a fixed line bundle, that is, a vector bundle of rank one (which means that the corresponding preprojective (or preinjective) $\Lambda$-module is of rank 1 (−1, respectively). The bundle $L$ plays the role of the structure sheaf; we always consider $\mathcal{H}$ together with $L$.

As mentioned above, over an algebraically closed field $k$ the situation is completely understood, we have $\mathcal{H} = \text{coh}(\mathbb{P}^1(k))$, the category of coherent sheaves over the projective line. The advantage of the algebraically closed case is the existence of a suitable explicitly given projective coordinate algebra, namely the polynomial algebra $\overline{k}[X, Y]$, graded by total degree. This is a factorial ring, in the graded sense, so that the points of its projective spectrum $X = \mathbb{P}^1(\overline{k})$ are in one-to-one correspondence with the irreducible homogeneous polynomials in $\overline{k}[X, Y]$ (up to multiplication with scalars). We will show that this can be generalized to arbitrary tame bimodules in a sense we will explain below.

If the field $k$ is not algebraically closed, the problem of explicitly describing the geometry is much harder and unsolved in general. Implicitly the geometry is given by the concept of the exceptional curves [67]. It is the aim of Chapters 1 and 2 to find a more explicit description for the geometry, and we will prove a structure theorem in full generality. Furthermore we will solve the problem completely for the field of real numbers (in Section 5.6), where the situation can be described in function-theoretical terms.

Let $k$ be an arbitrary field and $X$ be an exceptional curve over $k$ with associated hereditary category $\mathcal{H}$. In this general case we will show how to
construct projective coordinate algebras, though they will often be noncommutative. This is realized through the concept of an orbit algebra. Generally, given an object $A$ in a $k$-category $\mathcal{A}$ and some endofunctor $\sigma$ on $\mathcal{A}$, the orbit algebra with respect to $A$ and $\sigma$ is defined as

$$\Pi(A, \sigma) = \bigoplus_{n \geq 0} \text{Hom}_\mathcal{A}(A, \sigma^n A),$$

with multiplication given by the rule

$$g \ast f \overset{\text{def}}{=} \sigma^m(g) \circ f,$$

where $f \in \text{Hom}(A, \sigma^m A)$ and $g \in \text{Hom}(A, \sigma^n A)$. Formation of orbit algebras is a standard tool for obtaining projective coordinate algebras in algebraic geometry (although not under this name) and is also used in representation theory, see $[6, 64, 48]$. The general, categorical nature of the definition allows in particular to consider noncommutative situations. M. Artin and J. J. Zhang used orbit algebras to define noncommutative projective schemes $[2]$ and to prove an analogue of Serre’s theorem $[100]$.

In the following we restrict our attention to the homogeneous case. In this situation one can take for example the structure sheaf $A = L$ or a tilting object $A = T$ in $\mathcal{H}$ and for $\sigma$ the inverse Auslander-Reiten translation $\tau^-$. Then it is easy to see that the pair $(A, \tau^-)$ is a so-called ample pair ($[2, 103]$), and thus by the theorem of Artin-Zhang $[2$, Thm. 4.5$]$

$$\mathcal{H} \cong \frac{\text{mod}^Z(\Pi(A, \tau^-))}{\text{mod}^0_0(\Pi(A, \tau^-))},$$

the quotient category modulo the Serre subcategory of $\mathbb{Z}$-graded modules of finite length. Hence $\Pi(A, \tau^-)$ is a projective coordinate algebra for $X$, and it coincides with the preprojective algebra defined in $[6]$ (which for $A = L$ is also called the small preprojective algebra). However the graded algebras constructed in this way are often not practical for studying the geometry of $X$ explicitly. In the case of the Kronecker algebra we have $X = \mathbb{P}^1(k)$, the projective line over $k$ (understood in the scheme sense), and

$$\Pi(L, \tau^-) = k[X^2, XY, Y^2],$$

which consists of the polynomials in $X$ and $Y$ of even degree. Moreover, the orbit algebra $\Pi(T, \tau^-)$ with respect to $T$ is neither commutative nor a domain. Both algebras are projective coordinate algebras for $\mathbb{P}^1(k)$, as is the full polynomial algebra $k[X, Y]$, graded by total degree. This example illustrates the well-known fact that projective coordinate algebras are not uniquely determined, and also that there are projective coordinate algebras which are more useful than others. Of the three, only $k[X, Y]$ is graded factorial.
It is our aim to find graded factorial coordinate algebras in general. We will show that this is always possible, even given by an orbit algebra $\Pi(L, \sigma)$, for a suitable autoequivalence $\sigma$ on $\mathcal{H}$. Of course, one has to replace the usual factoriality by a noncommutative notion. We will then analyze the connection between the homogeneous prime spectrum of such a projective coordinate algebra for $X$ and the geometry of $X$. A similar analysis was done before in [6] for the preprojective algebras, but with less explicit results.

In the homogeneous case we are (up to tilting) in a situation which was studied in the cited papers by Dlab and Ringel, by Baer, Geigle and Lenzing, and by Crawley-Boevey. Over the real numbers the structure of $X$ as topological space is described explicitly in [23, 24, 25]. In [87, 28] and more generally in [17] an affine part of $X$ is described by the simple modules over a (not necessarily commutative) principal ideal domain. In [17] additionally a (commutative) projective curve is constructed, parametrizing the points of $X$, which is the centre of the noncommutative projective curves considered in [63] and [6]. A model-theoretic approach using the Ziegler spectrum is described by Prest [84] and Krause [50, Chapter 14]. But all these results describe the geometry of $X$ only partially.

In these notes we describe a link between the prime spectrum of certain projective coordinate algebras and the tubular shift automorphisms. The geometry of $X$ is given by the hereditary category $\mathcal{H}$. This means that understanding of the interplay between vector bundles and objects of finite length is important. In particular, with the structure sheaf $L$, for any point $x \in X$ and the corresponding simple object $S_x \in \mathcal{U}_x$ the bimodule $\text{End}(S_x) \text{Hom}(L, S_x) \text{End}(L)$ is of interest. By Serre duality this is equivalent to studying the bimodule $\text{End}(L) \text{Ext}^1(S_x, L) \text{End}(S_x)$, and this leads directly to the universal extension

$$0 \longrightarrow L \xrightarrow{\pi_x} L(x) \xrightarrow{e(x)} S_x \longrightarrow 0,$$

with the multiplicity (which was first introduced by Ringel in [88])

$$e(x) = [\text{Ext}^1(S_x, L) : \text{End}(S_x)].$$

These numbers can be arbitrarily large (Lemma 1.7.10). The above universal extension (for $L$) is a special case of a more general construction which leads to the tubular shift $\sigma_x$ of $\mathcal{H}$, sending an object $A$ to $A(x)$.

We would like to interpret the kernel $\pi_x$ (for each $x \in X$) as element in a suitable orbit algebra. This is accomplished by an automorphism $\sigma$ on $\mathcal{H}$ which we call efficient (in 1.1.3). Such an automorphism always exists and
has the property that the middle term $L(x)$ in the universal extension is of the form $L(x) \cong \sigma^d(L)$ for some positive integer $d$, depending on $x$.

The following theorem provides an explicit one-to-one correspondence between the points of $\mathcal{X}$ and the projective prime spectrum of $R$, given by forming universal extensions.

**Theorem.** Let $R = \Pi(L, \sigma)$ with $\sigma$ being efficient. Let $S_x$ be a simple sheaf concentrated in the point $x \in \mathcal{X}$. Let

$$0 \longrightarrow L \overset{\pi_x}{\longrightarrow} \sigma^d(L) \overset{S_x^e}{\longrightarrow} 0,$$

be the $S_x$-universal exact sequence of $L$. Then the element $\pi_x$ is normal in $R$, that is, $R\pi_x = \pi_x R$. Furthermore, $P_x = R\pi_x$ is a homogeneous prime ideal.

Moreover, for any homogeneous prime ideal $P \subset R$ of height one there is a point $x \in \mathcal{X}$ such that $P = P_x$.

See 1.2.3 and 1.5.1 for the complete statements.

Since a commutative noetherian domain is factorial if and only if each prime ideal of height one is principal, the following definition is natural. It is a graded version of a concept introduced by Chatters and Jordan [12]: a noetherian graded domain $R$, not necessarily commutative, is called a (noncommutative) graded factorial domain if each homogeneous prime ideal of height one is principal, generated by a normal element.

It follows that the concept of graded factoriality is also present in case of an arbitrary base field:

**Corollary.** Each homogeneous exceptional curve admits a projective coordinate algebra which is graded factorial.

We call the endomorphism ring of the unique generic $\Lambda$-module, which is a skew field, the function field. The importance of this invariant was demonstrated by Ringel in [88].

Since its introduction in [88] little was known about the multiplicities $e(x)$. In this article we will present some new results about their role. The conclusion is that they measure noncommutativity (“skewness”) in several senses:

**Theorem.** The function field of $\mathcal{X}$ is commutative if and only if the multiplicities $e(x) = 1$ for all $x \in \mathcal{X}$.

See Theorem 4.3.1 for the complete statement; it is also true for arbitrary exceptional curves and the commutative function fields are explicitly determined. Moreover:
• The multiplicities $e(x)$ are bounded from above by the square root $s(\mathcal{X})$ of the dimension of the function field over its centre. More precisely, if $e^*(x)$ denotes the square root of the dimension of $\text{End}(S_x)$ over its centre, then always $e(x) \cdot e^*(x) \leq s(\mathcal{X})$, and equality holds for all points $x$ except finitely many (Corollary 2.2.13 and Corollary 2.3.5).

• In our graded factorial algebra $R$ we have unique factorization in the sense that each normal homogeneous element is an (essentially unique) product of prime elements (which are by definition homogeneous generators of prime ideals of height one). In contrast to the commutative case, a prime element $\pi_x$ may be factorized into a product of several irreducible elements. The number of these factors is essentially given by $e(x)$ (see 1.6.5 and Proposition 1.6.6).

• We describe the localization $R_P$ at a prime ideal $P$. It turns out that $R_P$ is a local ring if and only if the corresponding multiplicity $e(x)$ is one; otherwise $R_P$ is not even semiperfect (Proposition 2.2.15).

Let us summarize these results: While we still have graded factoriality (in the non-commutative sense), several new ring theoretic effects appear.

Another new phenomenon due to noncommutativity is the occurrence of so-called ghost automorphisms. Denote by $\text{Aut}(\mathcal{X})$ the group of all automorphisms of the category $\mathcal{H}$ which leave the structure sheaf $L$ fixed. Let $R = \Pi(L, \sigma)$ be the orbit algebra formed with respect to an efficient automorphism $\sigma$. Every prime element $\pi_y \in R$ (that is, a normal element generating the prime ideal $P_y$ associated to the point $y$) induces a graded algebra automorphism $\gamma_y$ on $R$, given by the formula $r\pi_y = \pi_y \gamma_y(r)$. This in turn induces an automorphism $\gamma^*_y \in \text{Aut}(\mathcal{X})$ whose action on the set of all points of $\mathcal{X}$ is invisible, but it is a non-trivial element of $\text{Aut}(\mathcal{X})$ if and only if for all units $u$ the element $\pi_y u$ is not central. That is, in this case the functor $\gamma^*_y$ acts trivially on objects but non-trivially on morphisms. Such a functor we call a ghost automorphism.

The simplest example in which this effect arises is given by the bimodule $M = c(\mathbb{C} \oplus \overline{\mathbb{C}})_{\mathbb{C}}$ over $k = \mathbb{R}$, where $\mathbb{C}$ acts from the right on the second component via conjugation. For the associated curve $X$ a projective coordinate algebra is given by the graded twisted polynomial ring $R = \mathbb{C}[X; Y, \overline{\sigma}]$, graded by total degree, where $X$ is a central variable and for the variable $Y$ we have $Y a = \overline{\sigma} Y$ for all $a \in \mathbb{C}$. We write $R = \mathbb{C}[X, \overline{Y}]$. Then $Y$ is a prime element which is not central (up to units). It follows that complex conjugation induces a ghost automorphism of $\mathcal{X}$.

The following theorem expresses the interrelation between various automorphisms in more detail.
THEOREM. Let $R = \Pi(L, \sigma)$, where $\sigma$ is efficient. Let $\pi_y$ be a prime element of degree $d$ in $R$, associated to the point $y$ and $\gamma_y$ the induced graded algebra automorphism. Let $\sigma_y$ be the tubular shift associated to $y$. Then there is an isomorphism of functors $\sigma_y \simeq \sigma^d \circ \gamma_y^*$. 

We deduce the following:

- central prime elements $\pi_x$ behave well in the sense that the corresponding tubular shifts $\sigma_x$ are just given by a power of the degree shift $\sigma$.
- non-central prime elements $\pi_y$ lead to tubular shifts $\sigma_y$ which coincide with a power of the degree shift $\sigma$ only up to the ghost automorphism induced by $\pi_y$.

The theorem contains important information about the structure of the Picard group $\mathrm{Pic}(X)$, defined as the subgroup of $\mathrm{Aut}(\mathcal{H})$ generated by all tubular shifts $\sigma_x$ ($x \in X$). In particular, in contrast to the algebraically closed case, the Picard group need not be isomorphic to $\mathbb{Z}$.

In Chapter 5 we develop a technique which allows explicit calculation of the automorphism group $\mathrm{Aut}(X)$, provided we have knowledge of the Galois groups of the underlying skew field extensions $F/k$ and $G/k$, where $M = FM_G$ ($k$ acting centrally).

This will be illustrated for a special class of tame bimodules which are non-simple, generalizing the case $M = \mathbb{C} \oplus \mathbb{C}$: For $\alpha \in \mathrm{Aut}(F/k)$ denote by $M = M(F, \alpha)$ the bimodule $M = F \oplus F$ with $f \cdot (a, b) = (fa, fb)$ and $(a, b) \cdot f = (af, b\alpha(f))$ for all $f, a, b \in F$. Let $X$ be the associated exceptional curve. In this case, there is a point $x$ such that with the tubular shift $\sigma_x$ a graded factorial coordinate algebra is given by

$$\Pi(L, \sigma_x) = F[X; Y, \alpha],$$

where $X$ is central and $Y$ subject to the relation $Yf = \alpha(f)Y$. Denote by $\sigma_y$ the tubular shift associated to the point corresponding to the prime element $Y$. The ghost group is the subgroup of $\mathrm{Aut}(X)$ consisting of all ghost automorphisms. We have the following result on the automorphism group, the Picard group and the Auslander-Reiten translation.

THEOREM. Let $F$ be a skew field over $k$ such that the centre $Z$ of $F$ has cyclic Galois group $\mathrm{Gal}(Z/k)$ of order $r$, and assume that the restriction of $\alpha \in \mathrm{Aut}(F/k)$ to $Z$ is a generator of this group. Let $X$ be the homogeneous curve associated with the bimodule $M(F, \alpha)$.

1. If $r = 1$ then $\mathrm{Aut}(X)$ is canonically isomorphic to $\mathrm{PGL}_2(Z)$.
2. If $r \geq 2$ then $\mathrm{Aut}(X)$ is generated by
   - the automorphism $\alpha^*$ induced by $\alpha$, which generates the ghost group $G$ and is of order $r$;
• transformations of the form $Y \mapsto aY$ for $a \in \mathbb{Z}^*$; two transformations $Y \mapsto aY$ and $Y \mapsto bY$ (with $a, b \in \mathbb{Z}^*$) give the same automorphism on $X$ if and only if $a$ and $b$ have the same norm;
• the automorphism exchanging $X$ and $Y$, in case $r = 2$.

Moreover, the Picard group $\text{Pic}(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_r$, and for the Auslander-Reiten translation the following formula holds true:

$$\tau = \sigma_x^{-1} \circ \sigma_y^{-1} = \sigma_x^{-2} \circ \alpha^*.$$  

The last statement is surprising. On objects the Auslander-Reiten translation $\tau$ acts like $\sigma_x^{-2}$, which is the degree shift by $-2$. But on morphisms the ghost automorphism induced by $\alpha$ enters the game. In general the action of the Auslander-Reiten translation on morphisms has not been extensively studied and our result shows that interesting effects are to be expected.

So far in this introduction we have described the homogeneous case only. These notes also deal with the weighted case. Each non-homogeneous exceptional curve admits an underlying homogeneous curve. The following results show that the problem of determining the geometry of an exceptional curve can often be reduced to the homogeneous case.

• We show that insertion of weights into a central prime element in a graded factorial coordinate algebra preserves the graded factoriality; the resulting graded algebra is a projective coordinate algebra of a (weighted) exceptional curve (THEOREM 6.2.4).
• The automorphism group of a (weighted) exceptional curve is given by the automorphisms of the underlying homogeneous curve preserving the weights (PROPOSITION 6.3.1).

The last statement implies in particular that the ghost automorphisms of an exceptional curve are just those of the underlying homogeneous curve.

The insertion of weights is particularly important for our treatment of the tubular case in Chapter 8. The tubular exceptional curves have a strong relationship to elliptic curves. They are defined by the condition that the so-called virtual genus $g_X$ is one.

Considering tubular exceptional curves over arbitrary fields is of particular interest, since it is possible to construct tubular curves which are not isomorphic but derived equivalent, in other words, are Fourier-Mukai partners. We gave a first example for this in [55].

The main feature of the tubular case is that the derived category $\text{D}^b(X)$ consists entirely of tubular families. More precisely, there is the notion of a slope of non-zero objects in $\mathcal{H}$. For all $q \in \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ denote by $\mathcal{H}^{(q)}$ the full subcategory of $\mathcal{H}$ formed by the semistable objects of slope $q$. Then $\text{D}^b(X)$ is the additive closure of all $\mathcal{H}^{(q)}[n]$, where $(n, q) \in \mathbb{Z} \times \hat{\mathbb{Q}}$. The product $\mathcal{Y} = \mathbb{Z} \times \hat{\mathbb{Q}}$, endowed with the lexicographical order, is called the rational
helix. The automorphism group of $Y$ is isomorphic to the braid group $B_3$ on three strands [71].

Each category $\mathcal{H}^{(q)}$ is a tubular family, provided $\mathcal{H}^{(q)} \neq 0$ holds. The non-trivial fact that $\mathcal{H}^{(q)} \neq 0$ holds for all $q \in \hat{\mathbb{Q}}$ was proved in [55, 58] as part of a case by case analysis. We give a new proof here which is more systematic and therefore have an improved argument for the following statement.

**Proposition.** Let $X$ be a tubular exceptional curve. Then the automorphism group $\text{Aut}(\mathcal{D}^b(X))$ acts naturally on the rational helix and induces a homomorphism of groups

$$\Phi : \text{Aut}(\mathcal{D}^b(X)) \longrightarrow \text{Aut}(\mathcal{Y}) = B_3.$$ 

We point out how the kernel and the image of $\Phi$, and hence $\text{Aut}(\mathcal{D}^b(X))$ can be determined (8.1.9 and 8.1.10). The image of $\Phi$ is typically generated by the images of two or three so-called tubular mutations and the translation functor. It then can be decided essentially by the K-theory which tubular mutations to take for these generators. The kernel of $\Phi$ consists essentially of $\text{Aut}(X)$ and of the slope preserving elements of $\text{Pic}(X)$. Therefore we can apply the results from Chapter 5 on the determination of $\text{Aut}(X)$.

The number of orbits in $\mathbb{Z} \times \hat{\mathbb{Q}}$ under the above action is called the index of $X$. We proved in [58] that the index of a tubular exceptional curve is at most three and gave an example of index three. In other words, it is possible that there are up to three different tubular exceptional curves which are derived equivalent. We exhibit this example here in detail. We present another example of a tubular exceptional curve which has the same Grothendieck group as the preceding example, but which is of index two. This shows that the index of a tubular curve is not a K-theoretical invariant. In both examples we determine $\text{Aut}(\mathcal{D}^b(X))$ explicitly (8.3.1 and 8.4.1).

Another interesting effect is the occurrence of line bundles which are non-exceptional. In the algebraically closed case each line bundle $L$ over an exceptional curve $X$ is exceptional, that is, satisfies $\text{Ext}^1(L, L) = 0$. Over an arbitrary field this is also true for line bundles over a domestic exceptional curve (that is, when the virtual genus satisfies $g_X < 1$). But in the tubular case over an arbitrary base field non-exceptional line bundles may occur. We characterize the cases when they appear and show how they can be determined explicitly (Section 8.5).

For example, let $X$ be the tubular exceptional curve over $k = \mathbb{R}$ which arises by insertion of the weight $p = 2$ into the central prime element $X^4 + Y^4$ in $\mathbb{C}[X, Y]$. This yields again a graded factorial algebra (where the grading group has torsion $\mathbb{Z}_2$; see 6.2.4). It is easy to construct a non-exceptional line bundle over $X$ (8.5.1). This shows in particular that, in contrast to the
commutative case, it is not true in general that graded factoriality implies that all line bundles are obtained from the structure sheaf by degree shift.

In Chapter 7 we exhibit as examples two results concerning the concept of exceptional objects. The first is on the transitivity of the braid group action on the set of complete exceptional sequences over arbitrary fields (joint work with Meltzer [59]). In this case the result remained valid but the proof given over an algebraically closed field ([77]) had to be changed. (See Section 7.1.)

In contrast to this, the second problem fails to be true over an arbitrary field. Over an algebraically closed field exceptional curves are characterized by certain graded factorial algebras (which follows from [54, 66]). But Lenzing pointed out a counterexample, showing that this is wrong over the field $\mathbb{F}_2$ ([68]). (See Section 7.2.)

We have seen that several interesting and surprising phenomena occur when an arbitrary base field is allowed. Along the way, we will point out several interesting open problems. The following are particularly worth mentioning:

- Find graded factorial projective coordinate algebras for all weighted cases (by a suitable method of inserting weights also into non-central prime elements).
- Determine the ghost group in general. Describe the action of the Auslander-Reiten translation on morphisms in general.
- The function field $k(X)$ is always of finite dimension over its centre. Is the square root of this dimension always the maximum of the multiplicity function $e$? Describe each multiplicity $e(x)$ in terms of the function field.

These notes are based on the author’s Habilitationsschrift with the title “Aspects of hereditary representation theory over non-algebraically closed fields” accepted by the University of Paderborn in 2004. The present version contains several more recent results, in particular those concerning the multiplicities in Chapter 2. For clarity we have now divided the material into two parts. Part 1 (Chapters 1 to 5) deals with the homogeneous (= unweighted) case. Weighted curves are considered in Part 2 (Chapters 6 to 8). Certainly, the treatment of the homogeneous case can be considered as the essential part of these notes since the main new concepts and phenomena (efficient automorphisms, the verification of graded factoriality, the connection between the multiplicities and noncommutativity, the Picard group, the automorphism group, the occurrence and the nature of the ghosts) are best described in the homogeneous case and there is some general technique of weight reduction. We describe the transition to the weighted curves in Chapter 6 where we show that graded factoriality is preserved under insertion of weights, at least in many situations. On the other hand, we would like to
stress that there are also interesting new effects which occur only in weighted cases like the existence of non-exceptional line bundles and the occurrence of Fourier-Mukai partners in some of the tubular cases.

We assume that the reader is familiar with the language of representation theory of finite dimensional algebras. We refer to the books of Auslander, Reiten and Smalø [4], and of Ringel [89].

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CHAPTER 0

Background

In this preliminary chapter we describe the setting and present the background material from the literature which will be used later. The main parts of this work will start with Chapter 1. We recommend to browse through this chapter or even start reading the work with Chapter 1 and look up items here when necessary.

0.1. Notation

We work over an arbitrary field $k$. If not otherwise specified, all categories will be $k$-categories and all functors will be $k$-functors and covariant. If the isomorphism classes of objects in a category $\mathcal{C}$ form a set, then we call $\mathcal{C}$ small. (This is often called skeletally-small in the literature.) If $X$ is an object in $\mathcal{C}$ we write $X \in \mathcal{C}$ instead of $X \in \text{Ob}(\mathcal{C})$.

All rings and algebras are associative with identity. If not otherwise specified, by modules we mean right modules, and all modules are unitary. The category of all $R$-modules is denoted by Mod$(R)$. The full subcategory of finitely presented $R$-modules is denoted by mod$(R)$. Since we will only consider noetherian situations, these are just the finitely generated modules. If $R$ is an algebra graded by an abelian group $H$ we denote by Mod$^H(R)$ the category of $H$-graded $R$-modules; the morphisms are those of degree zero. The subcategory mod$^H(R)$ is similarly defined like in the ungraded situation.

0.2. One-parameter families, generic modules and tameness

In this section we briefly recall the notions of one-parameter families and tameness. Although we will not explicitly use these facts later in the text, they serve as main motivation.

In the representation theory of finite dimensional algebras certain modules often form sets with geometric structure. By the Tame and Wild Theorem of Drozd [31] (see also [16]) the indecomposable modules over a non-wild (= tame) finite dimensional algebra over an algebraically closed field $\overline{k}$ essentially lie in rational one-parameter families, that is, families indexed by (an affine part of) the projective line $\mathbb{P}^1(\overline{k})$. 

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0.2.1 (Affine one-parameter families). Let $A$ be a finite dimensional algebra over an algebraically closed field $\overline{k}$. Let $M$ be a $\overline{k}[T]$-$A$-bimodule which is free of finite rank as left $\overline{k}[T]$-module. Consider the associated functor $$F_M = - \otimes_{\overline{k}[T]} M : \text{mod}(\overline{k}[T]) \to \text{mod}(A).$$ For each $\lambda \in \overline{k}$ let $S_\lambda$ be the simple $\overline{k}[T]$-module $\overline{k}[T]/(T - \lambda)$. If all the images $F_M(S_\lambda)$ are indecomposable and pairwise non-isomorphic, then $\{F_M(S_\lambda)\}_{\lambda \in \overline{k}}$ is called an affine one-parameter family (of indecomposable modules).

0.2.2 (Tame algebras). Let $A$ be a finite dimensional algebra over an algebraically closed field $\overline{k}$. Then $A$ is called tame, if for each natural number $d$ almost all indecomposable $A$-modules of dimension $d$ lie in a finite number of affine one-parameter families, that is, given $d$ there are finitely many $\overline{k}[T]$-$A$-bimodules $M_i$, free of finite rank over $\overline{k}[T]$, such that all but finitely many indecomposable $A$-modules of dimension $d$ are isomorphic to $F_{M_i}(S_\lambda)$ for some $i$ and some $\lambda \in \overline{k}$.

0.2.3 (Generic modules). In the study of one-parameter families the concept of a generic module is important ([18], also [49]). An $A$-module $M$ is called generic [18], if it is indecomposable, of infinite length over $A$, but of finite length over its endomorphism ring. Note that for each affine one-parameter family, given by a functor $F_M$, a generic $A$-module is given by $F_M(\overline{k}(T))$, where $\overline{k}(T)$ is the field of rational functions in one variable.

Crawley-Boevey [18] has shown that, over an algebraically closed field, $A$ is tame if and only if for any natural number $d$ there is only a finite number of generic modules of endolength $d$. (In the latter case one also says that $A$ is generically tame. This notion makes sense over any field.) He showed that in this case the generic modules correspond to the one-parameter families.

0.2.4 (The Kronecker algebra). The Kronecker algebra $\Lambda$ over an algebraically closed field $\overline{k}$ provides the prototype of a tame algebra as well as of a one-parameter family. It is defined to be the path algebra of the quiver

$$\bullet \rightarrow \rightarrow \rightarrow \bullet$$

and is isomorphic to $\Lambda = \left(\begin{smallmatrix} \overline{k} & 0 \\ \overline{k} & \overline{k} \end{smallmatrix}\right)$, where $\overline{k}^2 = \overline{k} \oplus \overline{k}$ is considered as $\overline{k}$-$\overline{k}$-bimodule. The module category $\text{mod}(\Lambda)$, as well as its Auslander-Reiten quiver, has a particular simple shape, it is trisected

$$\text{mod}(\Lambda) = \mathcal{P} \lor \mathcal{R} \lor \mathcal{Q},$$

where $\mathcal{P}$ is the preprojective component, consisting of the Auslander-Reiten orbits of two projective indecomposables, $\mathcal{Q}$ is the preinjective component,
consisting of the Auslander-Reiten orbits of two injective indecomposables, and \( \mathcal{R} \) consists of the regular indecomposable modules, all lying in homogeneous tubes. One can say that \( \mathcal{P} \) and \( \mathcal{Q} \) form the discrete part of \( \text{mod}(\Lambda) \) and \( \mathcal{R} \) forms the continuous part, since the tubes are parametrized by the projective line \( \mathbb{P}^1(\overline{k}) \). Moreover, if one forms the category

\[
\mathcal{H} \overset{\text{def}}{=} \mathcal{Q}[-1] \vee \mathcal{P} \vee \mathcal{R}
\]

inside the bounded derived category of \( \text{mod}(\Lambda) \), then \( \mathcal{H} \) is equivalent to \( \text{coh}(\mathbb{P}^1(\overline{k})) \), the category of coherent sheaves over \( \mathbb{P}^1(\overline{k}) \).

The regular indecomposable modules of a fixed dimension form the (rational) one-parameter families for \( \Lambda \) (leave out one tube for an affine family). The regular part \( \mathcal{R} \) itself forms a so-called separating tubular family, which means that any morphism from a preprojective indecomposable to a preinjective indecomposable factors through any predescribed tube.

There is (up to isomorphism) precisely one generic \( \Lambda \)-module, given by the representation

\[
\overline{k}(T) \xrightarrow{1/T} \overline{k}(T),
\]

where \( \overline{k}(T) \) is the field of rational functions in one variable, which is the function field of \( \mathbb{P}^1(\overline{k}) \); the endomorphism ring of this generic module is also given by \( \overline{k}(T) \).

Let \( A \) be a tame \( \overline{k} \)-algebra. By Drozd’s theorem all one-parameter families for \( A \) are rational. In all known examples these parametrizations can be realized by a functor \( \text{mod}(\Lambda) \longrightarrow \text{mod}(A) \).

0.2.5. Over an arbitrary field \( k \) there is still no convenient definition of tameness. The definition of generically tameness makes sense over any field and has many advantages, but it does not capture the geometric flavour of one-parameter families. One should expect that an extension of Drozd’s Tame and Wild Theorem over arbitrary field \( k \) holds in the sense that, roughly speaking, the indecomposable finite dimensional modules over any non-wild finite dimensional \( k \)-algebra lie essentially in one-parameter families which are induced by (affine parts of) the exceptional curves in the sense of Lenzing [67], and there is some justification to call these one-parameter families in general rational. Since such a definition should be made properly in terms of the function (skew) field, we will not use this notion further in these notes; we here study just the one-parameter families given by the exceptional curves. The projective line is related to the Kronecker algebra, just as the exceptional curves are related (up to so-called weights) to the
tame bimodules \( M = FM_G \) and their associated hereditary algebras

\[
\Lambda = \begin{pmatrix} G & 0 \\ M & F \end{pmatrix},
\]

which were studied by Dlab and Ringel in several papers (for example [23, 87, 28], to name a few). Therefore the tame bimodules are of fundamental importance in the study of one-parameter families. Note that in general different one-parameter families for a fixed finite dimensional \( k \)-algebra may be induced by different tame bimodules over \( k \), as the discussion in Chapter 8 shows.

0.2.6 (The weighted case). In general one has deal with the so-called weighted case which leads to the study of the canonical algebras and to the weighted projective lines (as Ringel pointed out in his survey [91]). Over algebraically closed fields, the canonical algebras were defined by Ringel [89] and the weighted projective lines by Geigle and Lenzing [33]. Both definitions were later extended to arbitrary fields. In the case of the canonical algebras this was done by Ringel and Crawley-Boevey [90], in the case of the weighted projective lines by Lenzing [67] who called the more general objects exceptional curves. The canonical algebras can be characterized (up to some tilting equivalence) as the class of finite dimensional algebras admitting a separating tubular family [69]. These tubular families are parametrized by the exceptional curves. The tame bimodule algebras correspond to the subclass of finite dimensional algebras whose tubes are all homogeneous. So we call the tame bimodule case also the homogeneous or unweighted case, the general case also the weighted case.

By some general techniques (perpendicular calculus [34]), insertion of weights [67]) the general, weighted case can be reduced essentially to the homogeneous case. Therefore, main parts of this article are concerned with the homogeneous case.

0.3. Canonical algebras and exceptional curves

In this section we describe briefly the general class of finite dimensional algebras admitting a separating tubular family. This is the class of concealed canonical algebras, which contains the class of canonical algebras and the class of tame hereditary algebras, in particular tame bimodule algebras. These algebras have a geometric counterpart, the exceptional curves (which were called weighted projective lines when \( k \) is algebraically closed). These curves correspond to the concealed canonical algebras via tilting theory, and accordingly are derived equivalent to the corresponding algebra. Thus the study of (concealed) canonical algebras is more or less equivalent to the study of exceptional curves. Since we are interested in certain geometrical aspects
of algebras, we prefer in this paper the usage of the language and theory of the exceptional curves.

0.3.1 (Concealed canonical algebras). Let $k$ be a field and $\Sigma$ a finite dimensional $k$-algebra, which is assumed to be connected. Denote by $\text{mod}(\Sigma)$ the category of finitely generated right $\Sigma$-modules. Then $\Sigma$ is concealed canonical ([69], see also [102]) if and only if $\text{mod}(\Sigma)$ contains a sincere separating exact subcategory $\text{mod}_0(\Sigma)$. This means

- Exactness. $\text{mod}_0(\Sigma)$ is an exact abelian subcategory of $\text{mod}(\Sigma)$, which is stable under Auslander-Reiten translation $\tau = D \text{Tr}$ and $\tau^- = \text{Tr} D$
- Separation. Each indecomposable from $\text{mod}(\Sigma)$ belongs either to $\text{mod}_0(\Sigma)$ or to $\text{mod}_+(\Sigma)$, which consists of all $M \in \text{mod}(\Sigma)$ such that $\text{Hom}(\text{mod}_0(\Sigma), M) = 0$, or to $\text{mod}_-(\Sigma)$, which consists of all $N \in \text{mod}(\Sigma)$ such that $\text{Hom}(N, \text{mod}_0(\Sigma)) = 0$.
- Sincerity. For each non-zero $M \in \text{mod}_+(\Sigma)$ there is a non-zero morphism from $M$ to $\text{mod}_0(\Sigma)$ and for each non-zero $N \in \text{mod}_-(\Sigma)$ there is non-zero morphism from $\text{mod}_0(\Sigma)$ to $N$.
- Stability. Each projective module belongs to $\text{mod}_+(\Sigma)$ and each injective module to $\text{mod}_-(\Sigma)$.

0.3.2. The most prominent classes of examples are the following:

(1) The canonical algebras, as defined by Ringel and Crawley-Boevey in [90]. Actually, every concealed canonical algebra is tilting equivalent to a canonical algebra. A canonical algebra is defined to be the tensor algebra of a species

\[
\begin{array}{ccc}
D_1 & \overset{D_1}{\longrightarrow} & D_1 \\
\downarrow & & \downarrow \text{\scriptsize{FMG}} \\
D_2 & \overset{D_2}{\longrightarrow} & D_2
\end{array}
\begin{array}{ccc}
\cdots & \overset{\cdots}{\longrightarrow} & \cdots \\
\downarrow & & \downarrow \text{\scriptsize{FMG}} \\
D_t & \overset{D_t}{\longrightarrow} & D_t
\end{array}
\]

modulo certain relations (for details we refer to [90]). Here, $FMG$ is a tame bimodule (see 0.3.16 below), and there are $t$ arms, the $i$-th arm of length $p_i \geq 1$, and the $D_i$ are finite dimensional skew fields over $k$, with $k$ lying in their centres. Moreover, there are $F$-$D_i$-bimodule $U_i$ and $D_i$-$G$-bimodules $V_i$ ($k$ acting centrally) on the arrows starting in the source and ending in the sink, respectively.
(2) The tame hereditary algebras. In particular, the tame bimodule algebras (see 0.3.16 and 0.5.1). Actually, by the so-called insertion of weights [67], and by the perpendicular calculus [34], two processes which are inverse to each other, many problems for concealed canonical algebras can be reduced to the special class of tame bimodule algebras. (We will explain this in 0.3.16.)

0.3.3 (Separating tubular family). A sincere separating exact subcategory \( \text{mod}_0(\Sigma) \) defines a separating tubular family of stable tubes [90]: there is the coproduct of categories

\[
\text{mod}_0(\Sigma) = \coprod_{x \in X} \mathcal{U}_x,
\]

where \( \mathcal{U}_x \) are connected, uniserial length categories, containing neither non-zero projective nor non-zero injective modules. The full subcategory \( \mathcal{T}_x = \text{ind}(\mathcal{U}_x) \) of indecomposable objects in \( \mathcal{U}_x \) is called a stable tube. Moreover, each non-zero morphism from an object in \( \text{mod}_+(\Sigma) \) to an object in \( \text{mod}_-(\Sigma) \) factorizes through any prescribed tube \( \mathcal{U}_x \).

0.3.4 (Associated hereditary category). In the preceding coproduct, \( X \) is an index set, which is equipped with geometric structure. In [69] there is defined an associated hereditary abelian \( k \)-category \( \mathcal{H} \). Hereditary means that \( \text{Ext}^i_{\mathcal{H}}(-,-) = 0 \) for all \( i \geq 2 \). Roughly speaking, to construct \( \mathcal{H} \) one takes the union of \( \text{mod}_0(\Sigma) \) and \( \text{mod}_+(\Sigma) \) and forms inside the bounded derived category \( \text{D}^b(\text{mod}(\Sigma)) \) (see [36]) the closure of this union under all inverse shift automorphisms defined to tubes in \( \text{mod}_0(\Sigma) \). By the construction it is immediate that the categories \( \mathcal{H} \) and \( \text{mod}(\Sigma) \) are derived equivalent.

In the most important special case when \( \Sigma \) is a tame bimodule algebra we describe the category \( \mathcal{H} \) more explicitly in 0.5.1.

0.3.5 ( Bundles/objects of finite length). Denote by \( \mathcal{H}_0 \) (\( \mathcal{H}_+ \), respectively) the full subcategory of \( \mathcal{H} \) of objects of finite length (of objects, not containing objects \( \neq 0 \) of finite length, respectively). Then each indecomposable object in \( \mathcal{H} \) is either in \( \mathcal{H}_0 \) or in \( \mathcal{H}_+ \), and \( \text{Hom}_{\mathcal{H}}(\mathcal{H}_0, \mathcal{H}_+) = 0 \). The objects of \( \mathcal{H} \) (\( \mathcal{H}_+ \), respectively) are also called sheaves (vector bundles or torsionfree, respectively). By construction of \( \mathcal{H} \) we have \( \mathcal{H}_0 = \text{mod}_0(\Sigma) = \coprod_{x \in X} \mathcal{U}_x \).

0.3.6 (Exceptional curves). \( X \), together with the category \( \mathcal{H} \), is called an exceptional curve [67], and one sometimes writes \( \mathcal{H} = \text{coh}(X) \). This class of categories \( \mathcal{H} \) is characterized independently of the construction above by the following properties:

- \( \mathcal{H} \) is a connected small abelian \( k \)-category with finite dimensional morphism spaces.
• $\mathcal{H}$ is hereditary and noetherian and contains no non-zero projective object.

• $\mathcal{H}$ admits a tilting object (see the following number).

0.3.7 (Tilting object). $T \in \mathcal{H}$ is called a tilting object, if

• $\text{Ext}^1_{\mathcal{H}}(T, T) = 0$, and

• If $X \in \mathcal{H}$, then $\text{Hom}_{\mathcal{H}}(T, X) = 0 = \text{Ext}^1_{\mathcal{H}}(T, X)$ implies $X = 0$.

A tilting object lying in $\mathcal{H}^+$ is called a tilting bundle.

There exists even a tilting bundle $T$ such that $\text{End}_{\mathcal{H}}(T)$ is a canonical algebra (\cite[Prop. 5.5]{69}).

0.3.8 (Exceptional object). An object $E$ in $\mathcal{H}$ is called exceptional if it is indecomposable and $\text{Ext}^1_{\mathcal{H}}(E, E) = 0$. It follows then by an argument by Happel and Ringel \cite{40} that $\text{End}_{\mathcal{H}}(E)$ is a skew field.

0.3.9 (Serre duality). For an exceptional curve there is an autoequivalence $\tau$ on $\mathcal{H}$ such that Serre duality

$$\text{Ext}^1_{\mathcal{H}}(X, Y) \simeq D \text{Hom}_{\mathcal{H}}(Y, \tau X)$$

holds functorially in $X, Y \in \mathcal{H}$, where $D$ is the duality $\text{Hom}_{k}(-, k)$.

Since the category $\mathcal{H}$ is hereditary, the (bounded) derived category $D^b(\mathcal{X})$ is just the repetitive category of $\text{coh}(\mathcal{X})$. Moreover, $\mathcal{H}$ has almost split sequences and the Serre functor $\tau: \mathcal{H} \to \mathcal{H}$ serves as Auslander-Reiten translation. Denote by $\tau^-$ the inverse Auslander-Reiten translation.

0.3.10 (Grothendieck group). Denote by $K_0(\mathcal{X})$ the Grothendieck group of $\mathcal{H}$. Since $\mathcal{H}$ and $\text{mod}(\Sigma)$ have the same bounded derived category, we have $K_0(\mathcal{X}) = K_0(\Sigma)$, and this is a free abelian group of finite rank. We denote by $[X]$ the class in $K_0(\mathcal{X})$ of an object $X \in \mathcal{H}$.

$K_0(\mathcal{X})$ is equipped with the (normalized) Euler form $(-, -)$. This bilinear form is defined on classes of objects $X, Y$ in $\mathcal{H}$ by

$$\langle [X], [Y] \rangle = \frac{1}{m} (\dim_k \text{Hom}_{\mathcal{H}}(X, Y) - \dim_k \text{Ext}^1_{\mathcal{H}}(X, Y)),$$

where $m$ is a positive integer such that the resulting bilinear form maps onto $\mathbb{Z}$.

The Auslander-Reiten translation $\tau$ induces the Coxeter transformation, which we also denote by $\tau$ (by a slight abuse of notation), and which is an automorphism on $K_0(\mathcal{X}) = K_0(\Sigma)$ preserving the Euler form. The radical of $K_0(\mathcal{X})$ is defined by $\text{Rad}(K_0(\mathcal{X})) = \{ x \in K_0(\mathcal{X}) \mid \tau x = x \}$.

0.3.11 (Weights). For each $x \in \mathcal{X}$ let $p(x)$ be the rank of the tube $\mathcal{T}_x$. That is, $p(x)$ is the number of isomorphism classes of simple objects in $\mathcal{U}_x$. The tube $\mathcal{T}_x$, or the point $x$, is called homogeneous (\cite{89}), if $p(x) = 1,
exceptional otherwise. \( X \) is called homogeneous if all \( p(x) = 1 \). Clearly, a point \( x \) is exceptional if and only if a simple object \( S_x \) in \( U_x \) is exceptional.

Each exceptional curve admits only a finite number of exceptional points. Denote by \( x_1, \ldots, x_t \in X \) the exceptional points. We call the numbers \( p_i = p(x_i) > 1 \) weights, accordingly \((p_1, \ldots, p_t)\) the weight sequence.

0.3.12 (Rank). We define the rank of sheaves: Let \( x_0 \in X \), and let \( S_0 \) be a simple sheaf in the tube \( U_{x_0} \) of rank \( p_0 \). Let \( w := \sum_{j=0}^{p_0-1} [\tau^j S_0] \), which is an element of \( \text{Rad} K_0(X) \). By [69] we can assume that \( x_0 \) is a so-called rational point (see 0.4.4), that is, \( \mathbb{Z}w \) is a direct summand of \( \text{Rad} K_0(X) \).

After normalizing the linear form \( \langle -, w \rangle \) on \( K_0(X) \) by the factor \( c := [\mathbb{Z} : \langle K_0(X), w \rangle] \), we get a surjective linear form, compatible with the Coxeter transformation: For each \( x \in K_0(X) \) define \( \text{rk} x := \frac{1}{c} \langle x, w \rangle \), and moreover \( \text{rk}(X) = \text{rk}([X]) \) for each \( X \in \mathcal{H} \). Let \( X \in \mathcal{H} \) be indecomposable. Then \( \text{rk}(X) = 0 \) if and only if \( X \in \mathcal{H}_0 \); if \( X \in \mathcal{H}_+ \), then \( \text{rk}(X) > 0 \).

0.3.13 (Function field). The quotient category of \( \mathcal{H} \) modulo the Serre subcategory \( \mathcal{H}_0 \), formed by the objects of finite length, is equivalent to the category of finite dimensional vector spaces over some skew field which is (up to isomorphism) uniquely determined by \( X \). We call this skew field the function field. We denote it by \( k(X) = k(\mathcal{H}) \):

\[
\mathcal{H}/\mathcal{H}_0 \cong \text{mod}(k(X)).
\]

We call an exceptional curve \( X \) commutative if the function field \( k(X) \) is commutative.

The function field is known to be of finite dimension over its centre and to be an algebraic function (skew) field of one variable over \( k \) (in the sense of [104]), see [6].

If \( L \in \mathcal{H}_+ \) is a line bundle, that is, of rank one, then \( k(X) \) is isomorphic to the endomorphism ring of \( L \) considered as object in \( \mathcal{H}/\mathcal{H}_0 \) (given by fractions of morphisms of the same degree). Moreover, the rank of an object \( X \in \mathcal{H} \) agrees with the dimension of the vector space over \( k(X) \) corresponding to \( X \) considered as object in \( \mathcal{H}/\mathcal{H}_0 \).

The function field coincides with the endomorphism ring of the generic module associated with the separating tubular family \( \text{mod}_0(\Sigma) \) and was already studied in detail in [88].

0.3.14 (Special line bundle). From each of the exceptional tubes choose a simple sheaf \( S_i \in U_{x_i} \). Note that these simple sheaves are exceptional. In the following let \( L \in \mathcal{H}_+ \) be a line bundle, and assume additionally that for each \( i \in \{1, \ldots, t\} \) we have \( \text{Hom}(L, \tau^j S_i) \neq 0 \) if and only if \( j \equiv 0 \mod p_i \). Such a line bundle \( L \) exists by [69, Prop. 4.2] and is called special. It follows from [69, 5.2] that \( L \) is exceptional, since \( \text{End}_{\mathcal{H}}(L) \) is a skew field and \( a := [L] \)
is a root in $K_0(X)$. Recall from [65, 56] that $v \in K_0(X)$ is a root if $\langle v, v \rangle > 0$ and $\frac{\langle x, v \rangle}{\langle v, v \rangle} \in \mathbb{Z}$ for all $x \in K_0(X)$. For example, the class of an exceptional object is a root. Moreover, an exceptional object is uniquely determined (up to isomorphism) by its class.

In the sequel, we will always consider $\mathcal{H}$ together with a special line bundle $L$, also called a structure sheaf. Of course, if $X$ is homogeneous then each line bundle is special.

0.3.15 (Degree). Let $p$ be the least common multiple of the weights $p_1, \ldots, p_t$. Define $\langle \langle -,- \rangle \rangle := \sum_{j=0}^{p-1} \langle \tau^j-,- \rangle$ and define the degree function $\deg : K_0(X) \to \mathbb{Z}$ by

$$\deg x := \frac{1}{c} \left( \langle \langle a, x \rangle \rangle - \text{rk} \, x \langle \langle a, a \rangle \rangle \right),$$

where as above $a = [L]$.

0.3.16 (Underlying tame bimodule). Let $L$ be a special line bundle and $S_1, \ldots, S_t$ simple objects from the different exceptional tubes such that $\text{Hom}(L, S_i) \neq 0$. Let $S = \{ \tau^j S_i | 1 \leq i \leq t, j \not\equiv -1 \mod p_i \}$. Then the right perpendicular category $S^\perp$ is equivalent to $\text{mod}(\Lambda)$, where $\Lambda$ is a tame hereditary $k$-algebra of the form

$$\Lambda = \begin{pmatrix} G & 0 \\ M & F \end{pmatrix},$$

where $M = FM_G$ is a tame bimodule (also called affine bimodule), that is:

- $F$ and $G$ are skew fields, finite dimensional over $k$;
- $k$ lies in the centres of $F$ and $G$ and acts centrally on $M$.

We say that $M$ is a (tame) bimodule of (dimension) type $(2, 2), (1, 4)$ or $(4, 1)$ if this pair is $([M : F], [M : G])$. We call the number $\varepsilon \in \{1, 2\}$ the numerical type of $M$ (or of $X$), which is defined by

$$\varepsilon = \begin{cases} 
1 & \text{if } M \text{ is of type } (2, 2), \\
2 & \text{if } M \text{ is of type } (1, 4) \text{ or } (4, 1).
\end{cases}$$

The numerical type is an invariant of the curve $X$.

With $\kappa := \langle [L], [L] \rangle$, for the normalization factor $c = [\mathbb{Z} : (K_0(X), w)]$ as above we have $c = \kappa \varepsilon$.

0.3.17 (Automorphism groups). Let $X$ be an exceptional curve with associated abelian hereditary category $\mathcal{H}$ and structure sheaf $L$. Denote by $\text{Aut}(\mathcal{H})$ the automorphism class group of $\mathcal{H}$, that is, the group of isomorphism classes of autoequivalences of $\mathcal{H}$ (in the literature sometimes also called the Picard group [7], which has a different meaning in our presentation). We call this group the automorphism group of $\mathcal{H}$ and call the elements
automorphisms. (If there is need to emphasize the base field \(k\), we also write \(\text{Aut}_k(\mathcal{X})\) and use a similar notation in analogue situations.)

By a slight abuse of terminology, we will also call the autoequivalences themselves automorphisms, that is, the representatives of such classes; if \(F\) is an autoequivalence, then its class in the automorphism group is also denoted by \(F\).

The subgroup of elements of \(\text{Aut}(\mathcal{H})\) fixing \(L\) (up to isomorphism) is denoted by \(\text{Aut}(\mathcal{X})\), the automorphism group of \(\mathcal{X}\). (We will later see that this group does not dependent on \(L\).)

Each element \(\phi \in \text{Aut}(\mathcal{H})\) induces a bijective map \(\overline{\phi}\) on the points of \(\mathcal{X}\) by \(\phi(U_x) = U_{\overline{\phi}(x)}\) for all \(x \in \mathcal{X}\). We call \(\overline{\phi}\) the shadow of \(\phi\). If \(\phi\) lies in the kernel of the homomorphism \(\text{Aut}(\mathcal{H}) \rightarrow \text{Bij}(\mathcal{X})\), \(\phi \mapsto \overline{\phi}\), then we call \(\phi\) point fixing (or invisible on \(\mathcal{X}\)). If \(\overline{\phi}(x) = x\) we also say (by a slight abuse of terminology) that the point \(x\) is fixed by \(\phi\).

Denote by \(\text{Aut}_0(\mathcal{H})\) the (normal) subgroup of \(\text{Aut}(\mathcal{H})\) given by the point fixing automorphisms. Non-trivial elements of \(\text{Aut}(\mathcal{X})\) which are point fixing are called ghost automorphisms, or just ghosts. The subgroup \(\mathcal{G}\) of \(\text{Aut}(\mathcal{X})\) formed by the ghosts is called the ghost group. It is a normal subgroup of \(\text{Aut}(\mathcal{H})\). We have \(\mathcal{G} = \text{Aut}(\mathcal{X}) \cap \text{Aut}_0(\mathcal{H})\). We call the factor group \(\text{Aut}(\mathcal{X})/\mathcal{G}\) the geometric automorphism group of \(\mathcal{X}\), its elements geometric automorphisms. By a slight abuse of terminology, we also call the elements in \(\text{Aut}(\mathcal{X})\) which are not ghosts geometric.

Denote by \(\text{Aut}(\mathbb{D}^b(\mathcal{X}))\) the group of isomorphism classes of exact autoequivalences of the triangulated category \(\mathbb{D}^b(\mathcal{X})\), called the automorphism group of \(\mathbb{D}^b(\mathcal{X})\). (Compare also [8]. There is also the related notion of the derived Picard group [80].)

0.3.18 (Projective coordinate algebras). Let \(H\) be a finitely generated abelian group of rank one, which is equipped with a partial order \(\leq\), compatible with the group structure. Let \(R = \bigoplus_{h \in H} R_h\) be an \(H\)-graded \(k\)-algebra, such that each homogeneous component \(R_h\) is finite dimensional over \(k\) and such that \(R_h = 0\) for \(0 \not\leq h\). Assume moreover that \(R\) is a finitely generated \(k\)-algebra and noetherian. Note that we do not require that \(R\) is commutative.

Denote by \(\text{mod}^H(R)\) the category of finitely generated right \(H\)-graded \(R\)-modules, and by \(\text{mod}_0^H(R)\) the full subcategory of graded modules of finite length (which is equivalent to finite \(k\)-dimension). This is a Serre subcategory of \(\text{mod}^H(R)\), that is, it is closed under subobjects, quotients and extensions. The quotient category \(\text{mod}^H(R)/\text{mod}_0^H(R)\) is taken in the Serre-Grothendieck-Gabriel sense. We refer to [83].
Then the graded algebra $R$ is called a *projective coordinate algebra* for $X$ if there is an equivalence of categories

$$\mathcal{H} \cong \frac{\text{mod}^H(R)}{\text{mod}_0^H(R)}.$$ 

Each exceptional curve admits a projective coordinate algebra, even a $\mathbb{Z}$-graded one (see 6.2.1). Thus, in the terminology of [2], $\mathcal{H}$ is a (noncommutative) noetherian projective scheme.

Note that a projective coordinate algebra is not uniquely determined by $X$. One of the main aims of this article is to show that there is a projective coordinate algebra with “good” ringtheoretical properties.

### 0.4. Tubular shifts

One of the most important concepts we will use in these notes is that of shift automorphisms as developed in [69], which is a particular class of tubular mutations [70, 78, 79]. For the details we refer to [69]. Since we will also deal with the degree shift of graded objects, we will call a shift automorphism in the sense of [69] a *tubular shift* or just *shift associated to a point*.

0.4.1. Let $X$ be an exceptional curve with associated hereditary category $\mathcal{H}$ and tubular family $\mathcal{H}_0 = \bigoplus_{x \in X} \mathcal{U}_x$, with connected uniserial length categories $\mathcal{U}_x$ which are pairwise orthogonal. We fix a point $x \in X$ of weight $p(x)$. Let $S_x$ be a simple object in $\mathcal{U}_x$, denote by $\mathcal{S}_x$ additive closure of the Auslander-Reiten orbit of $S_x$, which consists of the semisimple objects from $\mathcal{U}_x$.

Let $M$ be an object. By the semisimplicity of the category $\mathcal{S}_x$, for the object

$$M_x = \bigoplus_{j=1}^{p(x)} \text{Ext}^1(\tau^j S_x, M) \otimes_{\text{End}(S_x)} \tau^j S_x,$$

there is a natural isomorphism of functors

(0.4.1) \quad \eta_M : \text{Hom}(\cdot, M_x) |_{\mathcal{S}_x} \cong \text{Ext}^1(\cdot, M) |_{\mathcal{S}_x},

which by the Yoneda lemma can be viewed as short exact sequence

$$\eta_M : 0 \longrightarrow M \overset{\alpha_M}{\longrightarrow} M(x) \overset{\beta_M}{\longrightarrow} M_x \longrightarrow 0$$

such that the Yoneda composition $\text{Hom}(U, M_x) \longrightarrow \text{Ext}^1(U, M)$, $f \mapsto \eta_M \cdot f$ is an isomorphism for each $U \in \mathcal{S}_x$. $\eta_M$ is called the $S_x$-universal extension (or sequence) of $M$. (If $p(x) = 1$, we also call it $S_x$-universal.) By means of the identification $\text{Hom}(\cdot, M_x) |_{\mathcal{S}_x} = \text{Ext}^1(\cdot, M) |_{\mathcal{S}_x}$ the assignment $M \mapsto M_x$
extends to a functor \( u \mapsto u_x \) for each \( u : M \to N \) such that \( u \cdot \eta_M = \eta_N \cdot u_x \). Then \( M_x \) (\( u_x \)) is called the fibre of \( M \) (of \( u \), resp.) in \( x \).

Similarly, let
\[
x_M = \bigoplus_{j=1}^{p(x)} \text{Hom}(\tau^j S_x, M) \otimes_{\text{End}(S_x)} \tau^j S_x.
\]
Then there is a natural isomorphism \( \gamma_M : \text{Hom}(-, x_M) \mid_{S_x} \to \text{Hom}(-, S) \mid_{S_x} \), which corresponds to a morphism \( \gamma_M : x_M \to M \), called \( S_x \)-universal.

0.4.2. Denote by \( N_x \) the full subcategory of \( \mathcal{H} \) consisting of objects \( M \) such that \( \text{Hom}(U_x, M) = 0 \). There is an autoequivalences \( \sigma_x : \mathcal{H} \to \mathcal{H} \) associated to the point \( x \) or the tube with index \( x \), therefore called tubular shift associated to \( x \), with the following properties:

1. For each \( M \in N_x \) the object \( \sigma_x(M) \) agrees with \( M(x) \). Moreover, if also \( N \in N_x \) and \( u \in \text{Hom}(M, N) \), then \( \sigma_x(u) \) agrees with the unique morphism \( u(x) \) making the following diagram commutative
\[
\begin{array}{ccc}
0 & \to & M & \xrightarrow{\alpha_M} & M(x) & \xrightarrow{\beta_M} & M_x & \to & 0 \\
\downarrow{u} & & \downarrow{u(x)} & & \downarrow{u_x} & & \downarrow{u_x} & & \\
0 & \to & N & \xrightarrow{\alpha_N} & N(x) & \xrightarrow{\beta_N} & N_x & \to & 0.
\end{array}
\]
More precisely, \( u(x) \) is already uniquely determined by commutativity of the left hand square.

2. Let \( 0 \to M \xrightarrow{f} M' \xrightarrow{g} C \to 0 \) be a short exact sequence such that \( M, M' \in N_x \) and \( C \simeq M_x \). Then there is a commutative exact diagram
\[
\begin{array}{ccc}
0 & \to & M & \xrightarrow{\alpha_M} & M(x) & \xrightarrow{\beta_M} & M_x & \to & 0 \\
\downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \\
0 & \to & M & \xrightarrow{f} & M' & \xrightarrow{g} & C & \to & 0.
\end{array}
\]
(In fact, the isomorphism \( \text{Hom}(C, M_x) \simeq \text{Ext}^1(C, M) \) implies the pullback diagram above. In this diagram, the map \( C \to M_x \) is monic, since its kernel is a subobject of \( M' \in N_x \). Since \( C \) and \( M_x \) have the same length, the map is also epic.)

3. \( M \in \mathcal{H}^+ \) implies \( \sigma_x(M) \in \mathcal{H}_+ \).

4. If \( y \neq x \) then there is a natural isomorphism \( \sigma_x \circ \sigma_y \simeq \sigma_y \circ \sigma_x \). On \( U_y \) the tubular shift \( \sigma_x \) acts functorially as the identity.

5. If \( M \in U_x \) then there is the exact sequence
\[
0 \to x M \xrightarrow{\gamma_M} M \to \sigma_x(M) \to M_x \to 0.
\]
\( \sigma_x \) acts on objects in \( \mathcal{U}_x \) like \( \tau^- \).

(Remark: Assume that \( x \in \mathbb{X} \) is homogeneous. Then it is not true in general that the tubular shift \( \sigma_x \) or the Auslander-Reiten translation \( \tau \) coincides with the identity functor on the homogeneous tube \( \mathcal{U}_x \). This will be shown in 5.4.2 and 5.4.3.)

(6) There is a natural transformation \( \varepsilon_x : 1_{\mathcal{H}} \longrightarrow \sigma_x \), coinciding on \( \mathcal{N}_x \) with \( \alpha \). This natural transformation is also denoted by \( 1_{\mathcal{H}} \xrightarrow{x} \sigma_x \).

(7) On \( K_0(\mathbb{X}) \), \( \sigma_x \) induces the automorphism

\[
y \mapsto y - \sum_{j=1}^{p(x)} \frac{\langle y, [\tau^j S_x] \rangle}{|\text{End}(S_x)|} [\tau^j S_x],
\]

where \( | - | \) denotes the dimension over \( k \).

0.4.3 (Multiplicity [88, 69]). As special case we have: Let \( L \) be a special line bundle with \( \text{Ext}^1(S_x, L) \neq 0 \). Then the \( S_x \)-universal extension of \( L \) has the shape

\[
0 \longrightarrow L \longrightarrow L(x) \longrightarrow S_x^{e(x)} \longrightarrow 0,
\]

with \( e(x) = [\text{Ext}^1(S_x, L) : \text{End}(S_x)] \). The number \( e(x) \) is called the multiplicity of \( x \). It does not dependent on the choice of the special line bundle \( L \). By Serre duality, \( e(x) \) coincides with \( [\text{Hom}(L, S_x) : \text{End}(S_x)] \). A point \( x \) is called multiplicity free if \( e(x) = 1 \). The exceptional curve \( \mathbb{X} \) is called multiplicity free ([88]) if \( e(x) = 1 \) holds for all \( x \in \mathbb{X} \).

0.4.4 (Index). With the notations as in the preceding number, the dimension \( f(x) = \frac{1}{\varepsilon} \cdot [\text{Ext}^1(S_x, L) : \text{End}(L)] \) is called the index of \( x \). (Recall, that \( \varepsilon \) denotes the numerical type of \( \mathbb{X} \).) A point \( x \) is called rational if \( f(x) = 1 \). Such a point always exists [69, Prop. 4.1]. We call a homogeneous point \( x \) unirational if \( e(x) = 1 = f(x) \). Such a point does not always exist, compare 0.6.1.

The product \( e(x) \cdot f(x) \) is denoted by \( d(x) \) and called the exponent of \( x \).

0.4.5 (Symbol). Let \( x_1, \ldots, x_t \in \mathbb{X} \) be the exceptional points with weights \( p_i = p(x_i) \), and let \( f_i = f(x_i) \) the index and \( d_i = d(x_i) \) the exponent of the point \( x_i \) \( (i = 1, \ldots, t) \). Let \( \varepsilon \) be the numerical type of \( \mathbb{X} \). Following [65] we call the matrix

\[
\sigma[\mathbb{X}] = \sigma_\infty[\mathbb{X}] = \begin{pmatrix}
    p_1, \ldots, p_t \\
    d_1, \ldots, d_t \\
    f_1, \ldots, f_t \\
\end{pmatrix}
\]

the symbol of \( \mathbb{X} \). (We make the convention, that rows of the form 1, 1, \ldots, 1 and the entry \( \varepsilon = 1 \) are omitted in the notation of the symbol.)

For a point \( x \in \mathbb{X} \) we call the numbers \( p(x), f(x) \) and \( e(x) \) (or \( d(x) \)) together also the symbol data of \( x \). For any simple object \( S_x \) concentrated in
let $x$ such that $\text{Hom}(L, S_x) \neq 0$ we have $[\text{Hom}(L, S_x) : k] = \varepsilon \cdot f(x) \cdot [\text{End}(L) : k]$ and $[\text{End}(S_x) : k] = \varepsilon \cdot f(x) \cdot p/p(x)$. Moreover, $\text{deg}(S_x) = f(x) \cdot \frac{p}{p(x)}$ with the least common multiple $p$ of $p_1, \ldots, p_t$.

The symbol of $X$ determines the Grothendieck group $K_0(X)$ uniquely up to isomorphism which preserves the Euler form. (The converse also holds if $X$ is domestic.) We refer to [65, 56].

0.4.6 ([69, S15]). Let $M, N \in \mathcal{H}_+$ be non-zero and $x \in X$ a point. Then for sufficiently large (positive) $n$,

(a) $\text{Hom}(M, \sigma^n_x(N)) \neq 0$.
(b) $\text{Hom}(\sigma^n_x(M), N) = 0$.

By Serre duality, one gets similar formulae for the extension spaces.

0.4.7 (The Picard group). Denote by $\text{Pic}(X)$ the subgroup of $\text{Aut}(\mathcal{H})$ generated by all tubular shifts $\sigma_x$ ($x \in X$) and call it the Picard group. It is always abelian. By $\text{Pic}_0(X)$ denote the subgroup of those elements of $\text{Pic}(X)$ of degree zero. That is, $\sigma \in \text{Pic}(X)$ is of degree zero if and only if the degree of $\sigma(L)$ is zero. By 0.4.5, $\text{deg}(L(x)) = d(x) \cdot p/p(x)$ for all $x \in X$, and it follows, that the definition does not depend on the choice of the structure sheaf $L$. Every torsion element from $\text{Pic}(X)$ lies in $\text{Pic}_0(X)$. (The converse is an open question in general.)

0.4.8. Let $x \in X$ be a point and $\phi \in \text{Aut}(\mathcal{H})$. The object $\phi(S_x)$ is simple, concentrated in a point $y \in X$. Then as elements in $\text{Aut}(\mathcal{H})$,

$$\sigma_y = \phi \circ \sigma_x \circ \phi^{-1}.$$ 

In particular, $\text{Pic}(X)$ is a normal subgroup of $\text{Aut}(\mathcal{H})$.

It is sometimes useful to have a stronger formulation: there is a natural isomorphism $\sigma_y \circ \phi \overset{\mu}{\longrightarrow} \phi \circ \sigma_x$, which is compatible with the natural transformations $\phi \overset{\phi_{\sigma_x}}{\longrightarrow} \phi \sigma_x$ and $\phi \overset{y_{\sigma_x}}{\longrightarrow} \sigma_y \phi$. One shows this by first considering for $M \in \mathcal{N}_x$ the $\phi$-image of the $S_x$-universal extension of $M$ on the one hand and the $S_y$-universal extension of $\phi(M)$ on the other hand, and then using 0.4.2 (2). In a second step the natural isomorphism on $\mathcal{N}_x$ obtained in this way will be extended to $\mathcal{H}$. (Compare the proof of 3.1.2 for further details.)

0.5. Tame bimodules and homogeneous exceptional curves

0.5.1 (Bimodule algebra). In this article we only consider bimodules $M = F \cdot M G$, where $F$ and $G$ are skew fields of finite dimension over $k$ and with $k$ lying in their centres, and such that $M$ is finite dimensional over $k$, with $k$ acting centrally. Such bimodules, finite dimensional over a finite subfield, are also called algebraic [87].
Each bimodule $M = F M G$ gives rise to a finite dimensional $k$-algebra

$$\Lambda = \begin{pmatrix} G & 0 \\ M & F \end{pmatrix}$$

which is hereditary. Moreover, this algebra is of tame representation type (that is, not of finite and not of wild type) if and only if $M$ is a tame bimodule, that is, $[M : F] \cdot [M : G] = 4$. In this case, the indecomposable regular modules lie in homogeneous tubes.

More precisely, in the tame case there is the trisection

$$\text{mod}(\Lambda) = P \lor R \lor Q,$$

that is, $\text{mod}(\Lambda)$ is the additive closure of $P$, $R$ and $Q$, where $P$ is the preprojective component, consisting of the Auslander-Reiten orbits of two indecomposable projective modules, dually $Q$ is the preinjective component and

$$R = \bigsqcup_{x \in X} U_x$$

consists of the regular modules, whose indecomposable summands lie in homogeneous tubes $U_x$, and thus $R = \text{mod}_0(\Lambda)$ is the separating tubular family.

0.5.2 (Associated hereditary category). In this tame bimodule case, the associated hereditary abelian $k$-category $H$ is constructed in a simple manner,

$$H = Q[-1] \lor P \lor R,$$

formed inside the bounded derived category $D^b(\text{mod}(\Lambda))$, and $H$ (together with $X$) is a homogeneous exceptional curve. Conversely, any homogeneous exceptional curve is obtained in this way.

0.5.3 (Homogeneous exceptional curves). A homogeneous exceptional curve $H$ is characterized by the following properties:

- $H$ is a connected small abelian $k$-category with finite dimensional morphism spaces.
- $H$ is hereditary and noetherian and contains no non-zero projective object.
- $H$ admits a tilting object.
- For each simple object $S \in H$ we have $\text{Ext}^1(S, S) \neq 0$.

The last condition precisely means that all tubes in $H_0$ are homogeneous.

0.5.4 (Structure sheaf, tilting bundle). In the homogeneous case any line bundle is special and can play the role of the structure sheaf $L$. Let $L$ be fixed. Let $\overline{L} \in H_+$ be indecomposable such that there is an irreducible morphism $L \rightarrow \overline{L}$. Then $M = \text{Hom}(L, \overline{L})$ is the (up to duality unique) underlying tame bimodule of $X$, and $\text{rk}(\overline{L}) = \varepsilon$ is the numerical type. Moreover, $T = L \oplus \overline{L}$ is a tilting bundle such that $\Lambda = \text{End}(T)$ is as in 0.5.1.
0.5.5 (The centre). Let $M$ be a tame bimodule and $\Lambda$ and $\mathcal{H}$ as above. Since the centre of $\mathcal{H}$ (or of $\Lambda$) is a field, it is sometimes useful to assume – without loss of generality – that $k$ is the centre of $\Lambda$. But we will not assume this in general.

The centre of $M$ is defined to be the set of all pairs $(f, g) \in F \times G$ such that $fm = mg$ for all $m \in M$. In this case, $f$ belongs to the centre of $F$ and $g$ to the centre of $G$, and the centre of $M$ is a field which can be identified with its projections into $F$ or into $G$, see [87, 5.2]. Of course, $K/k$ is a finite field extension, $M$ is a tame bimodule over $K$, and $\Lambda$ is a tame hereditary $K$-algebra with centre $K$.

Concerning dual bimodules there is the following general fact.

0.5.6 (Dual bimodule [21, 2.1.1]). Let $FM_G$ be an $F$-$G$-bimodule over $k$. There are isomorphisms of $G$-$F$-bimodules

$$\text{Hom}_F(FM_G, F_F) \simeq \text{Hom}_k(FM_G, k) \simeq \text{Hom}_G(FM_G, GG).$$

0.5.7. The index set $X$ above is naturally equipped with geometric structure, given by the hereditary category $\mathcal{H}$. It is the aim of the first part to study this structure. Whereas for algebraically closed field $\overline{k}$ this structure is well understood ($X$ is the projective line $\mathbb{P}^1(\overline{k})$) the structure in general can be very complicated. Unlike in the algebraically closed case it is in general even impossible to determine all the points of $X$. Also, there are points of many different kinds (different degrees, non-isomorphic endomorphism skew fields of the associated skyscraper sheaves,...)

One of the still easiest examples is the Kronecker algebra over $k = \mathbb{Q}$. Already in this innocent looking example the structure of $X$ is quite complicated. The points of $X = \mathbb{P}^1(\mathbb{Q})$ (in the scheme-sense) are in one-to-one correspondence with the irreducible homogeneous polynomials in $\mathbb{Q}[X, Y]$ (up to multiplication with non-zero scalars); in this ring there are infinitely many irreducible homogeneous polynomials in any degree; it is impossible to classify all these irreducible elements. Any finite field extension of $\mathbb{Q}$ occurs as endomorphism ring of a skyscraper sheaf.

0.6. Rational points

Let $X$ be an exceptional curve. In general it is hopeless to know all points of $X$. But often one has some control over the points “lying on the lowest level”, the rational points. Recall that for an exceptional curve $X$ rational points $x$, that means, with $f(x) = 1$, always exist (see 0.4.4).

Let $M = FM_G$ be a $(2,2)$-bimodule with associated homogeneous exceptional curve $X$ and $m$ a non-zero element in $M$. Then the representation

$$S_x = (F_F, G_G, \pi_m : F_F \otimes FM_G \simeq MG \longrightarrow MG/mG \simeq GG),$$
involving the canonical projection, induces a simple object in $\mathcal{H}$ concentrated in some point $x \in X$ (see [28]). Obviously, $x$ is a rational point, and each rational point arises in this way.

The following lemma is taken from [52, C.1]. It is a very useful tool for calculating multiplicities of rational points.

**Lemma 0.6.1.** Let $M = F M_G$ be a $(2, 2)$-bimodule. Let $m$ be a non-zero element in $M$. Let $x \in X$ be the induced rational point. For the multiplicity we have

$$e(x) = \frac{[F : k]}{[(F m \cap mG) : k]}.$$

If $M$ is a simple bimodule, or more generally, if $m$ is a bimodule generator of $M$, then $e(x) > 1$.

**Proof.** We have $\text{Ker}(\pi_m) = 1 \otimes mG$. Any endomorphism of $S_x$ is given by the commutative diagram

$$
\begin{array}{ccc}
F \otimes M & \xrightarrow{\pi_m} & G \\
| f & | & | \\
F \otimes M & \xrightarrow{\pi_m} & G,
\end{array}
$$

with $f \in F$ and $g \in G$, and it follows, that $fmG \subset mG$. Consider the subring $R = \{ f \in F \mid fmG \subset mG \}$ of $F$. Sending $(f, g)$ to $f$ yields an isomorphism $\text{End}(S_x) \simeq R$. Injectivity follows by applying $\pi_m$ to an element $1 \otimes y$, with $y \in M \setminus mG$. Surjectivity follows, since the map $f \cdot g$ for $f \in R$ restricts to the kernel of $\pi_m$, and hence induces a morphism $g \cdot \pi_m$ on $G$. Moreover, $f \mapsto fm$ gives rise to an isomorphism $R \simeq Fm \cap mG$, and with $e(x) = \frac{[\text{Hom}(L, S_x) : k]}{[\text{End}(S_x) : k]} = \frac{[\text{End}(L) : k]}{[\text{End}(S_x) : k]}$ the stated formula follows. Finally, if $m$ is a bimodule generator, we have $R \neq F$, since $M \neq mG$.

**Alternative proof.** We can consider $m$ as monomorphism between $L$ and $L$ (irreducible map) with cokernel $S = S_x$. Lifting endomorphisms from $\text{End}(S)$ to $\text{End}(L)$ induces an isomorphism between $\text{End}(S)$ and the subskewfield of $\text{End}(L)$ of those elements $g \in \text{End}(L)$ such that there is an $f \in \text{End}(L)$ such that $g \circ m = m \circ f$.

**Corollary 0.6.2.** Let $M = F M_G$ be a $(2, 2)$-bimodule. Then $M$ is a non-simple bimodule if and only if there exists a unirational point $x$.

**Proof.** If $M$ is simple then no unirational point exists by 0.6.1. If $M$ is non-simple then $F \simeq G$, without loss of generality $F = G$, and then $F M = F \oplus F$. By [87] there is an automorphism of $F$ over $k$ and an $(\alpha, 1)$-derivation $\delta$ of $F$ such that $(x, y) \cdot f = (xf + y\delta(f), y\alpha(f))$ for all $f, x, y \in F$. 

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Let $m = (1, 0)$. Then $Fm = mF$, hence by 0.6.1 the induced rational point has multiplicity one. □

**Lemma 0.6.3.** Let $M = {}_F M_G$ be a $(2, 2)$-bimodule. Let $m$ and $m'$ be non-zero elements in $M$ inducing points $x$ and $x'$, respectively. Then $x = x'$ if and only if there are non-zero elements $f \in F$ and $g \in G$ such that $m' = fmg$.

**Proof.** Assume $x = x'$. Let $S = S_x$. Consider the exact sequences

$$0 \to L \xrightarrow{m} L \xrightarrow{p} S \to 0,$$

$$0 \to L \xrightarrow{m'} L \xrightarrow{p'} S \to 0.$$

Applying the functor $\text{Hom}(L, -)$ to the lower sequence, since $\text{Ext}^1(L, L) = 0$ there is some $f \in \text{End}(L)$ such that $p' \circ f = p$, and this proves one direction. The converse is trivial. □
Part 1

The homogeneous case
CHAPTER 1

Graded factoriality

In this chapter we show how to associate with each homogeneous exceptional curve $X$ a (not necessarily commutative) graded factorial domain; it will be shown in the next chapter that such a factorial domain is a projective coordinate algebra for $X$. We use the term “graded factorial” for a graded version of rings which are called noncommutative noetherian unique factorization rings by Chatters and Jordan \cite{12, 46}.

Such a coordinate algebra will be constructed as orbit algebra $\Pi(L, \sigma)$ where $L$ is a line bundle and $\sigma$ a so-called efficient automorphism on $\mathcal{H}$. This means that $\sigma$ is point fixing such that the cyclic group $\langle \sigma \rangle$ acts on the set of line bundles “as transitively as possible”. This condition guarantees that the middle term in each $S_x$-universal sequence (defined in 0.4.1) of $L$

$$0 \longrightarrow L \xrightarrow{\pi_x} L(x) \longrightarrow S^e_x \longrightarrow 0$$

satisfies $L(x) \cong \sigma^d(L)$ for some natural number $d$ depending on $x$, and therefore the kernel $\pi_x$ can be interpreted as a homogeneous element in the orbit algebra $\Pi(L, \sigma)$. Note that this orbit algebra is noncommutative in general.

It is not difficult to see that for each homogeneous exceptional curve an efficient automorphism exists. The main result of this chapter is the following theorem (see 1.2.3 and 1.5.1).

**Theorem.** Let $R = \Pi(L, \sigma)$ with $\sigma$ being efficient. Let $S_x$ be a simple sheaf concentrated in the point $x \in X$. Let

$$0 \longrightarrow L \xrightarrow{\pi_x} \sigma^d(L) \longrightarrow S^e_x \longrightarrow 0,$$ 

be the $S_x$-universal exact sequence of $L$. Then the following conditions hold

1. The element $\pi_x$ is normal, that is, $R\pi_x = \pi_x R$.
2. $P_x = R\pi_x$ is a homogeneous prime ideal.
3. $P_x$ is a completely homogeneous prime ideal (that is, $R/P_x$ is a graded domain) if and only if $e = 1$.

Moreover, for any homogeneous prime ideal $P$ of height one there is a point $x \in X$ such that $P = P_x$.

Because of the last statement and since $R$ is also a noetherian domain, we say that $R$ is graded factorial, in analogy to commutative algebra.
With the theorem we have established a link between tubular shifts and the (projective) prime spectrum of \( R \). It turns out that graded factoriality is very useful for studying the geometry of \( X \).

### 1.1. Efficient automorphisms

Let \( X \) be a homogeneous exceptional curve with associated hereditary category \( \mathcal{H} \).

1.1.1. Recall that \( \text{Aut}_0(\mathcal{H}) \) is the subgroup of \( \text{Aut}(\mathcal{H}) \) consisting of those automorphisms (autoequivalences) \( \phi \) which are point fixing, that is, which satisfy \( \phi(S_x) \simeq S_x \) for all \( x \in X \). Note that for example the Auslander-Reiten translation \( \tau \), its inverse \( \tau^{-1} \) and all tubular shifts are in \( \text{Aut}_0(\mathcal{H}) \). We will usually assume (without loss of generality) that a point fixing automorphism \( \sigma \) of \( X \) (that is, a ghost) satisfies \( \sigma(A) = A \) (equality) for all objects \( A \in \mathcal{H}_0 \).

1.1.2. We fix a line bundle \( L \) (“structure sheaf”). Then \( L \) determines the degree function such that \( \text{deg}(L) = 0 \) (see 0.3.15). There is an indecomposable \( \overline{L} \in \mathcal{H}_+ \) such that there is an irreducible map \( L \longrightarrow \overline{L} \). Then \( T = L \oplus \overline{L} \) is a tilting bundle on \( \mathcal{H} \) such that \( \Lambda = \text{End}(T) \) is a tame hereditary bimodule algebra over \( k \) and the \( \text{End}(\overline{L}) \)-\( \text{End}(L) \)-bimodule \( M = \text{Hom}(L, \overline{L}) \) serves as underlying tame bimodule. The rank of \( \overline{L} \) coincides with the numerical type \( \epsilon \) of \( M \), hence is one or two. The Auslander-Reiten quiver (species) of \( \mathcal{H}_+ \) has the following shape:

\[
\cdots \quad \tau \overline{L} \longrightarrow \overline{L} \quad \cdots \quad \tau^{-1} \overline{L} \quad \cdots \quad \tau^{-2} \overline{L} \quad \cdots
\]

\[
\cdots \quad \tau L \quad \longrightarrow \quad L \quad \longrightarrow \quad \tau^{-1} L \quad \longrightarrow \quad \tau^{-2} L \quad \cdots
\]

where the dotted lines indicate the Auslander-Reiten orbits and \( M^* \) denotes the dual bimodule of \( M \). A line bundle is (up to isomorphism) uniquely determined by its degree. The precise value depends on whether \( \overline{L} \) is a line bundle or not. If it is a line bundle (that is, \( \epsilon = 1 \)) then we have \( \text{deg}(\tau^{-n}L) = 2n \) and \( \text{deg}(\tau^{-n}\overline{L}) = 2n + 1 \) for all \( n \in \mathbb{Z} \). If it is not a line bundle (that is, \( \epsilon = 2 \)), then \( \text{deg}(\tau^{-n}L) = n \) for all \( n \in \mathbb{Z} \).

**Definition 1.1.3** (Efficient automorphism). Let \( \sigma \in \text{Aut}(\mathcal{H}) \). We call \( \sigma \) **efficient** if it is point fixing and such that \( \text{deg}(\sigma L) > 0 \) is minimal with this property.

Obviously, if \( \sigma \) is efficient and \( \gamma \) is a ghost automorphism, then \( \gamma \circ \sigma \) and \( \sigma \circ \gamma \) are efficient.
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**Lemma 1.1.4.** Let $X$ be a homogeneous exceptional curve. Then there exists an efficient automorphism $\sigma$. Moreover, such an automorphism $\sigma$ is uniquely determined up to a ghost automorphism.

**Proof.** For the existence it is sufficient to remark that the inverse Auslander-Reiten translation $\tau^-$ is point fixing with $\deg(\tau^- L) = 2/\varepsilon > 0$. Thus there is an efficient automorphism $\sigma$ such that $1 \leq \deg(\sigma L) \leq 2/\varepsilon$. (Moreover, either $\sigma(L) \simeq \mathcal{L}$ or $\sigma(L) \simeq \tau^-(L)$. ) If $\sigma'$ is also efficient then $\sigma^{-1} \circ \sigma'$ fixes all objects in $\mathcal{H}$ and hence is a ghost automorphism. □

1.1.5 (The orbit cases). Here we present a division of tame bimodules. Each tame bimodule $M$ belongs to precisely one of the following three classes, called orbit cases:

I $M$ is a tame bimodule of type $(1,4)$ or $(4,1)$. In this case, the set of all line bundles coincides with the Auslander-Reiten orbit of $L$ and also with the $\mathrm{Aut}_0(\mathcal{H})$-orbit of $L$.

II $M$ is a tame bimodule of type $(2,2)$ and there is precisely one $\mathrm{Aut}_0(\mathcal{H})$-orbit of line bundles, that is, $\mathrm{Aut}_0(\mathcal{H})$ acts transitively on the set of all line bundles.

III $M$ is a tame bimodule of type $(2,2)$, and there are precisely two $\mathrm{Aut}_0(\mathcal{H})$-orbits of line bundles, coinciding with the Auslander-Reiten orbits.

Denote by $\mathcal{O}$ the $\mathrm{Aut}(\mathcal{H})$-orbit and by $\mathcal{O}_0$ the $\mathrm{Aut}_0(\mathcal{H})$-orbit of $L$, that is, $F \in \mathcal{H}$ lies in $\mathcal{O}_0$ if and only if there is $\sigma \in \mathrm{Aut}_0(\mathcal{H})$ such that $\sigma(L) \simeq F$. (Similarly for $\mathcal{O}$.)

**Remark 1.1.6.** In orbit cases I and III the inverse Auslander-Reiten translation $\sigma = \tau^-$ is an efficient automorphism. In orbit case II there is by definition a $\sigma \in \mathrm{Aut}_0(\mathcal{H})$ such that $\sigma(L) \simeq \mathcal{L}$, which gives an efficient automorphism. Moreover, by comparing dimensions of homomorphism spaces, $\sigma(\mathcal{L}) \simeq \tau^- L$ follows. Thus, in all orbit cases, if $\sigma$ is efficient, the cyclic group $\langle \sigma \rangle$ acts transitively on $\mathcal{O}_0$.

**Definition 1.1.7.** Let $\sigma \in \mathrm{Aut}(\mathcal{H})$. We call $\sigma$

- **positive**, if $\deg(\sigma L) > 0$.
- **exhaustive**, if the cyclic group $\langle \sigma \rangle$ acts transitively on $\mathcal{O}_0$.
- **transitive**, if $\langle \sigma \rangle$ acts transitively on $\mathcal{O}$.

**Lemma 1.1.8.** An autoequivalence $\sigma \in \mathrm{Aut}(\mathcal{H})$ is efficient if and only if it is positive, point fixing and exhaustive.

**Proof.** Follows immediately from Remark 1.1.6 by considering each of the three orbit cases. □

The following consequence is the main reason for defining efficient automorphisms and will be used in the next section.
COROLLARY 1.1.9. Let $\sigma$ be efficient and $\sigma_x$ be a tubular shift associated to a point $x$. Then there is some positive integer $d$ such that $\sigma_x(L) \simeq \sigma^d(L)$. 

Remark 1.1.10. (1) Assume that the underlying tame bimodule is non-simple of type $(2,2)$. Then there is a unirational point $x_0 \in X$. Let $\sigma_0$ be the corresponding tubular shift. Then $\langle \sigma_0 \rangle$ acts transitively on the set of isomorphism classes of line bundles, implying orbit case II.

(2) If $k$ is algebraically closed, or if $k = \mathbb{R}$, or if $k$ is a finite field, then each tame bimodule is either of orbit case I or non-simple as in (1).

(3) The bimodule $M = \mathbb{Q}(\sqrt{2})\mathbb{Q}(\sqrt{2},\sqrt{3})\mathbb{Q}(\sqrt{3})$ belongs to orbit case III. More generally each $(2,2)$-bimodule $F_MG$ with non-isomorphic $F$ and $G$ belongs to this class since there is no automorphism sending $L$ to $\mathcal{L}$.

(4) If $k$ is algebraically closed then an efficient automorphism $\sigma$ is uniquely determined. If $k = \mathbb{R}$ the same is true unless $M = \mathbb{C} \oplus \overline{\mathbb{C}}$; in that case we have the two possibilities $\sigma = \sigma_0$ as in (1) and $\sigma = \sigma_0 \circ \gamma = \gamma \circ \sigma_0$, where $\gamma$ is induced by complex conjugation. This will be proved in Section 5.3.

(5) In cases I and II a tubular shift $\sigma_x$ at a point $x$ is exhaustive (hence efficient) if and only if $x$ is a unirational point. In case III a tubular shift $\sigma_x$ is exhaustive if and only if either $f(x) = 1$ and $e(x) = 2$ or $f(x) = 2$ and $e(x) = 1$. Tubular shifts which are efficient do not always exist, see Example 1.1.13 below.

(6) There are also simple bimodules in orbit case II, see Example 1.1.13 below.

(7) We will see that in orbit case III for any point the product $e(x) \cdot f(x)$ is even (see 1.2.1 or 1.6.6).

The division of tame bimodules into the orbit cases will be very useful in the following. On the other hand, this division is quite formal. In order to get a better understanding it would be interesting to solve the following problem.

Problem 1.1.11. Find a criterion from which one can easily decide whether a given $(2,2)$-bimodule is of orbit case II or III.

We will later see that efficient automorphisms which are tubular shifts are advantageous for our considerations (see for example 1.7.1). In general efficient tubular shifts do not exist, see Example 1.1.13 below.

Problem 1.1.12. Does there always exist an efficient automorphism lying in the Picard group? (See the discussion in 3.2.12.)

Example 1.1.13. In [28, 5.3] the following type of $(2,2)$-bimodules is considered. Let $K$ be a (commutative) field with subfield $F$ and $G$, each of
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index 2 such that \( k = F \cap G \) is of finite index \( m \) in \( F \) (and \( G \)). Then let \( M \) be the \( F\)-\( G \)-bimodule \( K \).

1. If \( M \) is a simple bimodule if and only if \( F \neq G \). (Let \( F \neq G \). Assume that \( N \) is a proper, non-trivial subbimodule of \( M \). Then for \( 0 \neq n \in N \) we have \( Fn = N = nG \). Take an element \( f \in F \setminus G \) to get a contradiction.)

2. If \( m \) is odd then \( M \) is of orbit case II. This follows from 0.6.1 together with 1.1.10 (7).

3. If \( m \) is odd and \( F \neq G \) (hence \( m > 1 \)) then there is no efficient tubular shift. This follows from 1.1.10 (5) with 0.6.1.

4. An explicit example for which (3) holds is given as follows: Let \( k = \mathbb{Q} \) and \( M = \mathbb{Q}(\sqrt{2})\mathbb{Q}(\sqrt{2}, \zeta)\mathbb{Q}(\sqrt{2}) \) (where \( \zeta \) is a primitive third root of unity).

By 0.6.1 each rational point \( x \) has multiplicity \( e(x) = 3 \).

1.1.15 (Orbit algebra). Let \( \sigma \) be an efficient automorphism. Let \( R \) be the orbit algebra

\[
\Pi(L, \sigma) = \bigoplus_{n \geq 0} \text{Hom}_H(L, L(n)),
\]

where the multiplication is defined for elements \( r \in \text{Hom}(L, L(m)) \) and \( s \in \text{Hom}(L, L(n)) \) by \( s \ast r = \sigma^m(s) \circ r \). (We use the symbol \( \ast \) only if we would like to emphasize that this orbit algebra multiplication is meant.) This yields a positively \( \mathbb{Z} \)-graded \( k \)-algebra with \( R_0 = \text{End}(L) \) a skew field, and all homogeneous components \( R_n \) are finite dimensional over \( k \). Since non-zero morphisms between line bundles are monomorphisms, \( R \) is a graded domain, that is, for all non-zero homogeneous \( a, b \in R \) we have \( ab \neq 0 \).

For each \( n \geq 0 \) we have

\[
\dim_{R_0} R_n = \begin{cases} 
  n + 1 & \text{orbit case II} \\
  2n + 1 & \text{orbit cases I and III}.
\end{cases}
\]

In one formula: \( \dim_{R_0} R_n = \varepsilon \ell n + 1 \). Since each morphism between vector bundles is a sum of compositions of irreducible morphisms, \( R \) is generated in degrees 0 and 1. (This is also true if \( \sigma \) is not exhaustive.)
Note that for example in case $M = \mathbb{Q}(\sqrt{2}) \mathbb{Q}(\sqrt{2}, \sqrt{3}) \mathbb{Q}(\sqrt{3})$ the orbit algebras $\Pi(L, \sigma)$ and $\Pi(\overline{L}, \sigma)$ are not isomorphic as graded algebras.

For a $\mathbb{Z}$-graded algebra, we call the functor $\text{Mod}^\mathbb{Z}(R) \longrightarrow \text{Mod}^\mathbb{Z}(R)$, $X \mapsto X(1)$, the degree shift. For $R = \Pi(L, \sigma)$ this is induced by the automorphism $\sigma$, which we therefore also call the degree shift (see 2.1.6).

### 1.2. Prime ideals and universal extensions

We keep the notations from the previous section. In particular, $\sigma$ is efficient.

#### 1.2.1. Let $S$ be a simple object concentrated in the point $x \in X$. Let $e = e(x)$ be the multiplicity and $f = \deg(S)$ the degree of $S$ (compare 0.4.5). With the orbit number $\ell$ from 1.1.14 the quotient $d := ef/\ell$ is a natural number. It is easy to see (compare 1.1.9) that then $L(x) \cong L(d)$ (that is, $\sigma_x(L) \cong \sigma^d(L)$), and therefore the $S$-universal extension of $L$ is given by

$$0 \longrightarrow L \xrightarrow{\pi} L(d) \longrightarrow S^e \longrightarrow 0,$$

and the kernel $\pi$ is a homogeneous element in $R = \Pi(L, \sigma)$ of degree $d$.

**Lemma 1.2.2.** Let $\sigma$ be a positive and point fixing automorphism. Denote $\sigma^n(F)$ by $F(n)$. Consider the following diagram of exact sequences

$$0 \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{u_0} 0$$

$$0 \longrightarrow G_2 \xrightarrow{g_2} G_1 \xrightarrow{g_1} G_0 \xrightarrow{u_0} 0$$

with $F_1$, $F_2$, $G_1$, $G_2 \in \mathcal{H}_+$ and $F_0$, $G_0 \in \mathcal{H}_0$. Then there is some integer $n \geq 0$ and a commutative diagram

$$0 \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{u_0} 0$$

$$0 \longrightarrow G_2(n) \xrightarrow{g_2(n)} G_1(n) \xrightarrow{g_1(n)} G_0 \xrightarrow{u_0} 0$$

**Proof.** For $n \geq 0$ apply $\text{Hom}(F_1, -)$ to the short exact sequence

$$0 \longrightarrow G_2(n) \xrightarrow{g_2(n)} G_1(n) \xrightarrow{g_1(n)} G_0 \xrightarrow{u_0} 0.$$

Since by 0.4.6 for sufficiently large $n$ we have

$$\text{Ext}^1(F_1, G_2(n)) \cong D \text{Hom}(G_2(n), \tau F_1) = 0,$$

the map $\text{Hom}(F_1, G_1(n)) \longrightarrow \text{Hom}(F_1, G_0)$ is surjective, and the assertion follows immediately.
Of course, the lemma can be generalized in an obvious way to the weighted case.

Let $P$ be a (two-sided) homogeneous ideal in $R$. Then $P$ is called a (homogeneous) prime ideal, if for all $a, b$ homogeneous, $aRb \subset P$ implies $a \in P$ or $b \in P$. Moreover, $P$ is called a (homogeneous) completely prime ideal, if for all $a, b \in R$ homogeneous, $ab \in P$ implies $a \in P$ or $b \in P$. A homogeneous element $a$ in $R$ is called normal if $Ra = aR$. We additionally assume that normal and central elements are non-zero. If $R$ is a graded domain then a normal element $a$ defines a graded algebra automorphism $\gamma_a$ on $R$ by $ra = a\gamma_a(r)$ for all $r \in R$, and $a$ is central if and only if $\gamma_a = 1$.

**Theorem 1.2.3.** Let $R = \Pi(L, \sigma)$ with $\sigma$ being efficient. Let $S_x$ be a simple sheaf concentrated in the point $x \in X$. Let $e = e(x)$ be the multiplicity, $f = \deg(S_x)$ the degree of $S_x$, $d = ef/\ell$ and

$$0 \longrightarrow L \xrightarrow{\pi_x} L(d) \xrightarrow{s} S_x^e \longrightarrow 0,$$

the $S_x$-universal exact sequence of $L$. Then the following holds.

1. The homogeneous element $\pi_x$ is normal.
2. $P_x = R\pi_x$ is a homogeneous prime ideal.
3. $P_x$ is a completely homogeneous prime ideal if and only if $e = 1$.

**Proof.** We drop the index $x$ and write $S = S_x$, $\pi = \pi_x$ and $P = P_x$.

(1) Let $r \in R$ be homogeneous of degree $n$. We have the commutative diagram with (universal) exact sequences

$$0 \longrightarrow L \xrightarrow{\pi} L(d) \xrightarrow{s} S_x^e \longrightarrow 0,$$

for some $s$ (by universality 0.4.2 (1)). Since $\sigma$ is an equivalence there is some homogeneous $t \in R$ such that $s = t(d)$. Then, by the definition of the multiplication in $R$ we get $\pi r = t\pi$. Hence $\pi R \subset R\pi$. The reverse inclusion follows since each homogeneous component is finite dimensional.

(2) Let $P'$ be the graded ideal in $R$, whose homogeneous elements are given by those $r$ such that $rx = 0$, where $rx$ is given by the following diagram

$$0 \longrightarrow L \xrightarrow{\pi} L(d) \xrightarrow{r'} S_x^e \longrightarrow 0$$

($r$ homogeneous of degree $n$) where $r'$ is given as in (1) by universality. Obviously, $\pi \in P'$. Moreover, if $r \in P'$ is homogeneous of degree $n$, then
there is a commutative, exact diagram

\[
\begin{array}{cccccc}
0 & \to & L & \xrightarrow{\pi} & L(d) & \xrightarrow{p} & S^e & \to & 0 \\
& \downarrow{r} & \downarrow{r'} & & \downarrow{r_x=0} \\
0 & \to & L(n) & \xrightarrow{\pi(n)} & L(d+n) & \xrightarrow{p(n)} & S^e & \to & 0.
\end{array}
\]

Hence the zero morphism \(0 : S^e \to L(d+n)\) satisfies \(r_x = p(n) \circ 0\). By the Homotopy-Lemma [44, Lemma B.1] (applied to this special situation) there is an \(s \in \text{Hom}(L(d), L(n))\) such that \(r = s \pi\). Hence \(P' = R \pi\) follows.

Note that \(\text{End}(S^e) \cong M_e(D)\) (where \(D = \text{End}(S)\) is a skew field) is a prime ring and \(\sigma\) induces an automorphism of this ring. Using the formula \((s \ast r)_x = \sigma^m(s_x) \circ r_x\) (where \(m\) is the degree of \(r\)) it is sufficient to show that for some \(n \geq 0\) the map

\[
R_n \to \text{End}(S^e), \ r \mapsto r_x
\]

is surjective. But this follows from Lemma 1.2.2.

(3) Let \(e = 1\). If \(a, b \in R\) are homogeneous such that \(ab \in P\) then \((ab)_x = 0\). Since \(\text{End}(S)\) is a skew field, \(a_x = 0 \) or \(b_x = 0\), hence \(a \in P\) or \(b \in P\) and \(P\) is completely prime. For the converse, if \(e > 1\), then there are non-zero matrices \(A, B \in M_e(\text{End} S)\) such that \(A \cdot B = 0\). By the proof of (2) there are homogeneous \(a, b \in R\) such that \(b_x = B\) and \(a_x = \sigma^{-m}(A)\) (where \(m\) is the degree of \(b\)). It follows that \(ab \in P\), but \(a \notin P\) and \(b \notin P\). Hence \(P\) is not completely prime. \(\square\)

**1.3. Prime ideals as annihilators**

In this section we give another description of the homogeneous prime ideals which occur in Theorem 1.2.3. We assume \(R = \Pi(L, \sigma)\), where \(\sigma\) is efficient. As usual, we set \(F(n) = \sigma^n(F)\) for all \(F \in \mathcal{H}\).

1.3.1 (Fibre map). Let \(S\) be simple, concentrated in \(x\), let \(e = e(x)\).

For an \(f \in \text{Hom}(L, L')\), where \(L'\) is some line bundle, we have the following commutative diagram with universal exact sequences

\[
\begin{array}{cccccc}
0 & \to & L & \xrightarrow{\pi} & L(d) & \xrightarrow{f} & S^e & \to & 0 \\
& \downarrow{f} & \downarrow{f'} & & \downarrow{f_x} \\
0 & \to & L' & \xrightarrow{\pi'} & L'(d) & \xrightarrow{f} & S^e & \to & 0,
\end{array}
\]

with fibre map \(f_x\).

1.3.2 (1-irreducible maps). Let \(f\) be a (non-zero) morphism between line bundles. Then \(f\) is called 1-\textit{irreducible}, if whenever \(f = gh\) with morphisms
g and h between line bundles, then g or h is an isomorphism. The following facts are obvious:

1. Each non-zero map between line bundles has a factorization into 1-irreducible maps.
2. A morphism between line bundles is 1-irreducible if and only if its cokernel is a simple object.
3. Each simple object is cokernel of a 1-irreducible map. Moreover, one of the line bundles can be chosen arbitrarily.
4. If \( u : L \to L(n) \) is a 1-irreducible map, then it is an irreducible element in \( R \). The converse does not hold in general, in orbit case III.

The following lemma is a fundamental statement on 1-irreducible maps.

**Lemma 1.3.3.** Let \( S \) be simple, concentrated in \( x \), let \( \pi = \pi_x \) and \( e = e(x) \), and let
\[
0 \to L \xrightarrow{u} L' \to S \to 0
\]
be exact, where \( L' \) is a line bundle. Then there is a morphism \( v \in \text{Hom}(L', L(d)) \) such that \( \pi = vu \). Moreover, the fibre \( u_x : S^e \to S^e \) of \( u \) has kernel and cokernel isomorphic to \( S \).

**Proof.** Since \( \text{Hom}(S, S^e) \simeq \text{Ext}^1(S, L) \) by (0.4.1), there is a commutative exact diagram
\[
0 \to L \xrightarrow{\pi} L(d) \xrightarrow{\text{id}} S^e \to 0
\]
which proves the first part. For the fibre maps we have \( 0 = \pi_x = v_x u_x \). If \( u_x \) would be an epimorphism, then we would have \( v_x = 0 \). Then, as in the proof of part (2) of Theorem 1.2.3 we would get a non-zero \( s \in \text{Hom}(L'(d), L(d)) = 0 \), contradiction. Hence, the cokernel of \( u_x \) is non-zero, hence isomorphic to \( S \). By the snake lemma, the same follows for the kernel. \( \square \)

**Proposition 1.3.4.** Let \( S \) be simple, concentrated in the point \( x \), and \( P \) be the corresponding homogeneous prime ideal (by Theorem 1.2.3). Let \( M \) be the graded left \( R \)-module \( \bigoplus_{n \geq 0} \text{Ext}^1_H(S, L(n)) \). Then \( P = \text{Ann}_R(M) \).

**Proof.** Let \( r \in R_n, r \neq 0 \). The \( S \)-universal extension induces a commutative exact diagram
\[
0 \to L \xrightarrow{\pi} L(d) \xrightarrow{\text{id}} S^e \to 0
\]
\[
0 \to L(n) \xrightarrow{r} X \xrightarrow{\text{id}} S^e \to 0.
\]
If \( r \in \text{Ann}_R(M) \) then the lower sequence splits and \( r \in P \) follows immediately. For the converse, we show more generally the next proposition. \( \square \)

**Proposition 1.3.5.** Let \( S \) be a simple sheaf concentrated in \( x \) and \( P = R\pi \) be the corresponding homogeneous prime ideal. For each \( n \in \mathbb{N} \) let \( S^{(n)} \) be the indecomposable sheaf of length \( n \) with socle \( S \). Let \( M^{(n)} \) be the graded left \( R \)-module \( \bigoplus_{i \geq 0} \text{Ext}^1(S^{(n)}, L(i)) \). Then \( \text{Ann}_R(M^{(n)}) \supset R\pi^n = P^n \).

**Proof.** There is a short exact sequence

\[
0 \longrightarrow S \xrightarrow{i} S^{(n)} \xrightarrow{p} S^{(n-1)} \longrightarrow 0,
\]

which induces a short exact sequence of graded modules

\[
0 \longrightarrow M^{(n-1)} \xrightarrow{p^*} M^{(n)} \xrightarrow{i^*} M \longrightarrow 0.
\]

We have to show that \( \pi M^{(n)} \subset p^*(M^{(n-1)}) (\simeq M^{(n-1)}) \). A homogeneous, non-zero element \( \eta \) in \( M^{(n)} \) induces the following pushout diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & L \\
\uparrow & \downarrow \pi & \downarrow \pi \eta \\
\eta : 0 & \longrightarrow & X \\
\uparrow & \downarrow & \downarrow \\
\pi \eta : 0 & \longrightarrow & S^{(n)} \\
\uparrow & \downarrow & \downarrow \\
0 & \longrightarrow & S^{e}
\end{array}
\]

If \( X \) decomposes, \( X = L' \oplus E \) with \( E \neq 0 \) of finite length and \( L' \) a line bundle, then \( E \simeq S^{(i)} \) for some \( 1 \leq i \leq n \) (since \( S^{(n)} \) is uniserial), and we get the following commutative exact diagram

\[
\begin{array}{ccc}
E & \longrightarrow & S^{(i)} \\
\downarrow & \downarrow & \downarrow \\
\eta : 0 & \longrightarrow & L \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & L'
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & L' \oplus E \\
\downarrow & \downarrow & \downarrow \\
0 & \longrightarrow & S^{(n-i)}
\end{array}
\]
Then $\eta \in p^*(M^{(n-1)})$ follows. Similarly, if $X'$ decomposes, then $\pi \eta \in p^*(M^{(n-1)})$ follows. But if $X$ and $X'$ are indecomposable, hence line bundles, then the middle vertical short exact sequence is (up to shift) the $S$-universal sequence (for $L$ or for $\mathcal{T}$). It then follows that $a_x$ is an isomorphism. Since $a$ is a product of $n$ morphisms between line bundles with cokernel $S$, we get a contradiction by Lemma 1.3.3.

\[\square\]

**Corollary 1.3.6.** For each $x \in X$ denote by $P_x$ the homogeneous prime ideal as in Theorem 1.2.3. For each infinite subset $U$ of $X$,

\[\bigcap_{x \in U} P_x = 0.\]

**Proof.** Denote by $S_x$ the simple sheaf concentrated in $x$. Let $r \in R$ be non-zero and homogeneous of degree $n$. Choose $x \in U$ such that the cokernel in the short exact sequence

\[0 \longrightarrow L \overset{r}{\longrightarrow} L(n) \longrightarrow C \longrightarrow 0,\]

has no non-zero summand which is concentrated in $x$. Denote by $M_x$ the graded $R$-module $\oplus_{n \geq 0} \text{Ext}^1(S_x, L(n))$. Then $r \notin \text{Ann}_R(M_x) = P_x$ follows by the Homotopy-Lemma.

\[\square\]

### 1.4. Noetherianness

Let $R = \Pi(L, \sigma)$, where $\sigma$ is positive. We show that $R$ is noetherian. Since this basic property is very important we give a detailed proof. The arguments are mainly taken from [6].

1.4.1. Let $\mathcal{L}$ be the full subcategory of $\mathcal{H}$ formed by all $L(n)$ (where $n \in \mathbb{Z}$). Denote by $\mathcal{L}_+$ the full subcategory formed by all $L(n)$ (where $n \geq 0$). Similarly, denote by $\mathcal{L}_-$ the full subcategory formed by all $L(n)$ (where $n \leq 0$). We denote by $\text{Mod} \mathcal{L}$ the category of (covariant) $k$-functors $F: \mathcal{L} \rightarrow \text{Mod}(k)$ (similarly for $\mathcal{L}_+$ and $\mathcal{L}_-$), and by $\text{Mod}^Z(R)$ the category of $\mathbb{Z}$-graded right $R$-modules, by $\text{Mod}^Z(R)$ those graded modules $M$ with $M_n = 0$ for $n < 0$. The following obvious lemmas are proved as in [6, 3.6].

**Lemma 1.4.2** (covariant functors = left graded modules). There is an equivalence of $k$-categories

\[\text{Mod}(\mathcal{L}_+) \rightarrow \text{Mod}^Z(R^{\text{op}}), \ F \mapsto \bigoplus_{n \geq 0} F(L(n)).\]

For $X \in \mathcal{H}$ denote by $(X, -)$ the functor $\text{Hom}_{\mathcal{H}}(X, -)_{|\mathcal{L}_+}$, by $\text{Ext}^1_{\mathcal{H}}(X, -)$ the functor $\text{Ext}^1_{\mathcal{H}}(X, -)_{|\mathcal{L}_+}$. 

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**Lemma 1.4.3** (contravariant functors = right graded modules). There is an equivalence of $k$-categories

$$\text{Mod}(\mathcal{L}^{op}) \longrightarrow \text{Mod}^{\mathbb{Z}_+}(R), \quad F \mapsto \bigoplus_{n \geq 0} F(L(-n)).$$

For $X \in \mathcal{H}$ denote by $[-, X]$ the (contravariant) functor $\text{Hom}_\mathcal{H}(-, X)_{|\mathcal{L}}$, by $\text{Ext}_{\mathcal{H}}^1[-, X]$ the functor $\text{Ext}_{\mathcal{H}}^1(-, X)_{|\mathcal{L}}$.

**Proposition 1.4.4.** $R = \Pi(L, \sigma)$ is (graded) noetherian (left and right).

**Proof.** (1) (compare [6, 4.2]) We have to show that the functor $(L, -)$ is noetherian. For this it is sufficient to show, that $(L, -)/U$ is noetherian for each non-zero cyclic subfunctor $U$. Let $n \in \mathbb{Z}$ such that there is an epimorphism

$$(L(n), -) \longrightarrow U.$$

Then there is a non-zero morphism

$$\eta: (L(n), -) \longrightarrow (L, -)$$

induced by some morphism $f: L \longrightarrow L(n)$. We get a short exact sequence

$$0 \longrightarrow L \xrightarrow{f} L(n) \longrightarrow C \longrightarrow 0,$$

where $C \in \mathcal{H}_0$, which induces an exact sequence

$$0 \longrightarrow (C, -) \longrightarrow (L(n), -) \longrightarrow (L, -) \longrightarrow \text{Ext}_{\mathcal{H}}^1(C, -).$$

Hence it suffices to show that the functor $\text{Ext}_{\mathcal{H}}^1(S, -)$ is noetherian for each simple object $S \in \mathcal{H}_0$. But this follows precisely as in [6, 4.1]: we have to show that $\text{Ext}_{\mathcal{H}}^1(S, -)/U$ is noetherian for each non-zero cyclic (or finitely generated) subfunctor $U$. Let $(L(m), -) \longrightarrow \text{Ext}_{\mathcal{H}}^1(S, -)$ be a non-zero morphism with image $U$. This is induced by some non-split short exact sequence

$$0 \longrightarrow L(m) \longrightarrow L' \longrightarrow S \longrightarrow 0,$$

with some line bundle $L'$. We get an exact sequence

$$0 \longrightarrow (L', -) \longrightarrow (L(m), -) \longrightarrow \text{Ext}_{\mathcal{H}}^1(S, -) \longrightarrow \text{Ext}_{\mathcal{H}}^1(L', -).$$

By Serre duality and by the positivity of the grading of $R$ we see, that the functor $\text{Ext}_{\mathcal{H}}^1(S, -)$ is of finite length, therefore the same holds true for $\text{Ext}_{\mathcal{H}}^1(S, -)/U$.

(2) Similarly, we have to show that the contravariant functor $[-, L]$ is noetherian. As above, it suffices to show that $[-, L)/V$ is noetherian for each non-zero cyclic (contravariant) subfunctor $V$. Let $n \in \mathbb{Z}$ be such that there is an epimorphism

$$[-, L(-n)) \longrightarrow V.$$
Then there is a non-zero morphism

\[ [-, L(-n)) \rightarrow [-, L] \]

induced by some morphism \( g : L(-n) \rightarrow L \). We get a short exact sequence

\[ 0 \rightarrow L(-n) \xrightarrow{g} L \rightarrow B \rightarrow 0, \]

with \( B \in \mathcal{H}_0 \), hence an exact sequence

\[ 0 \rightarrow [-, L(-n)) \rightarrow [-, L) \rightarrow [-, B). \]

It suffices to show that \([-, S)\) is noetherian for each simple sheaf. Again we show that \([-, S)/V \) is noetherian for each non-zero cyclic (contravariant) subfunctor \( V \). We get a morphism

\[ [-, L(m)) \rightarrow [-, S) \]

with image \( V \). This induces a short exact sequence

\[ 0 \rightarrow L' \rightarrow L(m) \rightarrow S \rightarrow 0, \]

with some line bundle \( L' \), hence an exact sequence

\[ 0 \rightarrow [-, L') \rightarrow [-, L(m)) \rightarrow [-, S) \rightarrow \operatorname{Ext}^1[-, L'). \]

By Serre duality \( \operatorname{Ext}^1[-, L') \) is of finite length, hence we get the result. \( \square \)

**Proposition 1.4.5.** \( R = \Pi(L, \sigma) \) has graded (classical) Krull dimension two. That is, since \( R \) is a graded local domain, the only homogeneous prime ideals are the zero ideal, the homogeneous maximal (left) ideal \( \mathfrak{m} = \bigoplus_{n \geq 1} R_n \) and the homogeneous prime ideals of height one.

**Proof.** Let \( \mathcal{F} = \operatorname{mod}(\mathcal{L}_+), \) and denote by \( \mathcal{F}_0 \) the full subcategory of \( \mathcal{F} \) of objects of finite length. Denote by \( \mathcal{F}_1 \) the Serre subcategory of objects in \( \mathcal{F} \) which become of finite length in \( \mathcal{F}/\mathcal{F}_0 \). One shows as in [6, 4.6] that for each \( n \in \mathbb{Z} \) the functor \( (L(n), -] \) becomes simple in \( \mathcal{F}/\mathcal{F}_1 \). It follows (compare [76, 6.4.5]), that the classical Krull dimension of \( R \) is two. \( \square \)

**1.5. Prime ideals of height one are principal**

Let \( \mathcal{X} \) be a homogeneous exceptional curve. For \( x \in \mathcal{X} \) denote by \( P_x \) the homogeneous prime ideal of height 1 as in Theorem 1.2.3.

**Theorem 1.5.1.** Let \( R = \Pi(L, \sigma) \) with \( \sigma \) being efficient. Let \( P \) be a homogeneous prime ideal in \( R \) of height one. Then \( P = P_x \) for some \( x \in \mathcal{X} \).
Proof. Let \( a \in P \) be a non-zero homogeneous element. Let \( U \) be the subfunctor of \((L, -)\) corresponding to the graded module \( Ra \). There is an epimorphism \((L(m), -) \to U\). This induces a short exact sequence
\[
0 \to L \xrightarrow{a} L(m) \to C \to 0,
\]
where \( C \) is a coproduct \( \coprod_{i=1}^t S_i^{(n_i)} \), with (not necessarily non-isomorphic) simple \( S_1, \ldots, S_t \) in \( H_0 \), concentrated in \( x_1, \ldots, x_t \), respectively. Let \( M_i^{(n_i)} \) be the graded left \( R \)-module \( \bigoplus_{n \geq 0} \text{Ext}^1(S_i^{(n_i)}, L(n)) \) and \( P_i = R\pi_i \) be the homogeneous prime ideal corresponding to \( x_i \). We have \( P_i^{n_i} \subset \text{Ann}_R(M_i^{(n_i)}) \).

By applying \((-,-)\) to the short exact sequence (1.5.1) we get an exact sequence
\[
0 \to (L(m), -) \to (L, -) \to \text{Ext}^1(C, -).
\]
Thus \( R/Ra \) embeds into \( \bigoplus_{i=1}^t M_i^{(n_i)} \). Hence we get (using 1.3.5)
\[
P = \text{Ann}_R(R/P) \supset \text{Ann}_R(R/Ra) \supset \text{Ann}_R\left(\bigoplus_{i=1}^t M_i^{(n_i)}\right) \supset \bigcap_{i=1}^t R\pi_i^{n_i}.
\]
It follows that the product of some powers of the normal elements \( \pi_1, \ldots, \pi_t \) is in \( P \), hence \( \pi_i \in P \) for some \( i \). But then \( P = P_i \) follows. \( \square \)

Combining Theorem 1.2.3 and Theorem 1.5.1 we get the following important result.

Corollary 1.5.2. There is a natural bijection \( x \mapsto R\pi_x \) between points \( x \in X \) and homogeneous prime ideals \( P \subset R \) of height one, given by forming universal extensions. Under this bijection points of multiplicity one correspond to homogeneous completely prime ideals of height one. \( \square \)

Invoking Krull’s principal ideal theorem, a commutative noetherian integral domain is factorial (that is, a unique factorization domain) if and only if each prime ideal of height one is principal ([75, 20.1]). Inspired by this we make the following definition (where we allow abelian grading groups):

Definition 1.5.3. Let \( R \) be a noetherian graded domain. Then \( R \) is called a (noncommutative) graded factorial domain if each homogeneous prime ideal of height one is principal, generated by a normal element.

In our setting a graded factorial domain is nothing but the graded version of a noncommutative noetherian unique factorisation ring (UFR) in the sense of Chatters and Jordan [12, 46], where we here restrict our considerations to domains.

Note that on the other hand (besides the grading) the concept of a noncommutative unique factorization domain (UFD) in [11], where only completely prime ideals of height one are considered, is too restrictive for our
purposes. (Actually we will show that the rings occurring in our setting are (graded) UFD’s in the sense of Chatters [11] only when they are commutative, see Theorem 1.2.3 (3) and Theorem 4.3.5.) In his definition of unique factorization domains Cohn [13, 15] focussed on irreducible elements (atoms) rather than on prime elements/ideals.

Corollary 1.5.2 has the following important consequence.

**Corollary 1.5.4.** The orbit algebra \( \Pi(L, \sigma) \), where \( \sigma \) is efficient, is a (noncommutative) graded factorial domain. □

**Corollary 1.5.5.** In orbit cases I and III, the small preprojective algebra \( \Pi(L, \tau^-) \) is graded factorial.

**Proof.** In these cases the inverse Auslander-Reiten translation \( \tau^- \) is efficient. □

### 1.6. Unique factorization

Let \( X \) be homogeneous. Let \( R = \Pi(L, \sigma) \) be an orbit algebra where \( \sigma \) is an efficient automorphism.

1.6.1 (Prime and irreducible elements). Recall that a non-zero, homogeneous element \( a \in R \) is called normal if \( Ra = aR \) holds. We call a (non-zero) homogeneous element \( \pi \) in \( R \) prime if it is normal and if it generates a homogeneous prime ideal (necessarily of height one). A non-zero homogeneous element \( u \) is called irreducible if \( u = ab \) with \( a, b \) homogeneous, implies that either \( a \) or \( b \) is a unit.

We have a weak form of Euclid’s Lemma: If \( \pi \) is prime and divides the product \( ab \), where \( a \) and \( b \) are homogeneous, such that \( a \) or \( b \) is normal, then \( \pi \) divides \( a \) or \( b \).

**Lemma 1.6.2.** Let \( \pi_1, \pi_2 \) be two prime elements. Then there is an \( \alpha \in R^*_0 \) such that \( \pi_1 \pi_2 = \alpha \pi_2 \pi_1 \).

**Proof.** This follows from the properties of universal extensions, but there is also a purely ring theoretical proof. The assertion is clear if \( R \pi_1 = R \pi_2 \). Assume that \( R \pi_1 \neq R \pi_2 \). Since \( \pi_1, \pi_2 \) are normal, we have \( \pi_1 \pi_2 = \gamma(\pi_2) \pi_1 \), where \( \gamma \) is a degree preserving automorphism on \( R \). Then \( \gamma(\pi_2) \) is also prime and \( \pi_2 \) divides \( \gamma(\pi_2) \pi_1 \), hence \( \pi_2 \) divides \( \gamma(\pi_2) \), thus there is an \( \alpha \in R^*_0 \) such that \( \gamma(\pi_2) = \alpha \pi_2 \). □

**Proposition 1.6.3.** Let \( R = \Pi(L, \sigma) \) with \( \sigma \) being efficient. Each non-zero normal element of \( R \) is a product of prime elements. This factorization is unique up to permutation and multiplication with units.

**Proof.** Let \( a \neq 0 \) be a normal non-unit. By the principal ideal theorem (see [76, 4.1.11]) there is a prime ideal \( P_1 \) of height one such that \( a \in \)
There is a prime element $\pi_1$ such that $P_1 = R\pi_1$. Hence there is a homogeneous element $r_1$ such that $a = r_1\pi_1$. Since $r_1$ is normal (see below), the assertion follows because of the positivity of the grading by induction. The uniqueness (up to a unit) follows from the preceding lemma.

We show that $r_1$ is normal. For each normal element $f$ let $\gamma_f : R \rightarrow R$ be the automorphism such that $sf = f\gamma_f(s)$ for each $s \in R$. Let $s \in R, r := r_1, \pi := \pi_1$. Then 

$$(sr)\pi = s(r\pi) = sa = a\gamma_a(s) = r\pi_a(s) = r\pi^{-1}_a\gamma_a(s)\pi,$$

hence $sr = r\pi^{-1}_a\gamma_a(s)$, hence $Rr \subset rR$. The converse inclusion follows by $rs = \gamma_a^{-1}_a\gamma_a(s)r$ for each $s \in R$.

COROLLARY 1.6.4 (Almost commutativity of normal elements). Let $f_1, f_2$ be two normal elements. Then there is a homogeneous unit $\alpha \in R_0^*$ such that $f_1f_2 = \alpha f_2f_1$.

One can summarize the preceding results by saying that the non-zero normal elements modulo units form a UF-monoid in the sense of [15, Ch. 3], see also [46]. Moreover, as in [46, Prop. 2.2] it follows that each non-zero homogeneous ideal contains a normal element. ($R$ is said to be (graded) conformal.)

Obviously, each homogeneous element $r \in R, r \neq 0$, is a product of irreducible elements. If a normal element is irreducible it is prime.

1.6.5 (Ringtheoretic meaning of $e(x)$ and $f(x)$). Let $u$ be a homogeneous element of $R$ with cokernel $S$. Obviously, if $S$ is simple, then $u$ is irreducible. The converse also holds in orbit cases I and II, that is, here irreducible elements and 1-irreducible maps are the same concept. Hence, with the notations of Theorem 1.2.3, in orbit cases I and II the prime element $\pi = \pi_x$ is a product of $e = e(x)$ irreducible elements: $\pi = u_1u_2 \ldots u_e$. Moreover, each $u_i$ is of degree $f = f(x) = \deg(S)$.

In orbit case III we get a slightly different result, since then there are irreducible elements (even of degree one in $R$) with cokernel of length two; they may have two different points as support.

PROPOSITION 1.6.6. With the same notations as in Theorem 1.2.3, assume orbit case III. Then the following holds for $\pi = \pi_x$.

1) If $f = \deg(S)$ is even, then $\pi = u_1u_2 \ldots u_e$, where all $u_i \in R$ are irreducible of degree $f/2$ with cokernels isomorphic to $S$.

2) If $f = \deg(S)$ is odd, then $\pi = u_1u_2 \ldots u_t$, where $t = e/2$ and all $u_i \in R$ are irreducible of degree $f$ with cokernels isomorphic to $S^2$.

PROOF. A chain of projections $S^e \rightarrow S^{e-1} \rightarrow \ldots \rightarrow S$ yields a factorization $\pi = v_1v_2 \ldots v_e$ with $v_i : L^{(e-i)} \rightarrow L^{(e-i+1)}$, where $L^{(i)}$ are line bundles.
such that $L^{(0)} = L$ and $L^{(e)} = L(d)$. Moreover, the cokernels of the $v_i$ are isomorphic to $S$, hence $\deg L^{(i+1)} = \deg L^{(i)} + f$. If $f$ is even, all $L^{(i)}$ lie in the same orbit, whereas in case $f$ is odd, precisely the $L^{(i)}$, where $i$ is even, lie in the same orbit as $L$. So, in the first case, all $u_i := v_i$ are in $R$ and irreducible. In the second case, the elements $u_i = v_{2i-1}v_{2i}$ are in $R$ and irreducible. □

Note that in orbit case III there are also irreducible elements in $R$ with indecomposable cokernel $S_x^{(2)}$ of length two.

The following is a modification of Lemma 1.3.3 so that there is some left and right symmetry:

**Lemma 1.6.7.** Let $S$ be simple, concentrated in $x$, and let $u \in R$ be irreducible such that

$$0 \longrightarrow L \xrightarrow{u} L(f) \longrightarrow S \longrightarrow 0$$

is exact. Then there is a morphism $v' \in R_{d-f}$ such that $\pi_x = uv'$.

**Proof.** Let $\pi = \pi_x = ab$ be a product of homogeneous elements. Then the cokernel of $a$ (or $b$, resp.) is of the form $S^f$ for some $0 \leq f \leq d$. Then, by universality, there is a $b'$ such that $\pi = b'a$, and since $b'\pi = \pi b$, we get $b' = \gamma(b)$, where $\gamma : R \longrightarrow R$ is the automorphism such that $\pi r = \gamma(r)\pi$ for each $r \in R$. Now apply Lemma 1.3.3 to $\gamma^{-1}(u)$. □

**Remark 1.6.8.** There is also a version for irreducibles with cokernel $S^2$.

The same argument shows:

**Lemma 1.6.9.** Let $\pi = u_1 \ldots u_t$ be a factorization of the prime element $\pi$ into irreducibles $u_1, \ldots, u_t$. Let $\gamma : R \longrightarrow R$ be the automorphism such that $\pi r = \gamma(r)\pi$ for all $r \in R$. Then

$$\begin{align*}
\pi &= \gamma(u_t)u_1 \ldots u_{t-1} = u_t\gamma^{-1}(u_1)\ldots\gamma^{-1}(u_{t-1}) \\
&= \gamma(u_{t-1})\gamma(u_t)u_1 \ldots u_{t-2} = u_{t-1}u_t\gamma^{-1}(u_1)\ldots\gamma^{-1}(u_{t-2}) \\
&\vdots \\
&= \gamma(u_2)\ldots\gamma(u_t)u_1 = u_2\ldots u_t\gamma^{-1}(u_1).
\end{align*}$$

Accordingly, for all $i$ and $j$ there is an irreducible decomposition of $\pi$ where $u_i$ appears at position $j$. □

We remark that the behaviour of irreducible elements is not well-understood. For example, it is not true in general, that a prime element remains the same under each permutation of its irreducible factors. It may happen, that under some permutation of the factors the cokernel is not semisimple. Moreover, in orbit case III it is even unknown whether each irreducible element is divisor of a normal element.
Example 1.6.10. Let $\beta = e^{2\pi i/3}$ the third root of unity, let $k = \mathbb{Q}(\beta)$ and $K = k(t)$ with $t = \sqrt[3]{2}$. Let $R = K[X;Y,\alpha]$, where $\alpha$ is the $k$-automorphism on $K$ given by $t \mapsto \beta t$. (We will see in the next section that $R$ can be realized as orbit algebra $\Pi(L, \sigma)$ over a suitable exceptional curve with efficient automorphism $\sigma$.) Then
\[
\pi = Y^3 - 2X^3 = (Y - tX)(Y - \beta tX)(Y - \beta^2 tX)
\]
is a factorization of the central prime element $\pi$ into irreducibles. Then
\[
(Y - \beta tX)(Y - tX)(Y - \beta^2 tX) = Y^3 - 3\beta tXY^2 + 3\beta^2 tX^2Y - 2X^3,
\]
and the cokernel of this element is not semisimple, since it is not associated to $\pi$.

1.7. Examples of graded factorial domains

In this section we discuss some classes of examples of orbit algebras $\Pi(L, \sigma)$ with efficient $\sigma$. It follows in particular that these explicitly given algebras are graded factorial domains.

One main reason for preferring efficient automorphism which are tubular shifts (if they exist) when forming the orbit algebra is the following lemma (which is some converse of Theorem 3.1.2 below). It ensures the existence of central prime elements which is very useful for calculating examples (see also the proof of Theorem 4.3.5).

Lemma 1.7.1. Let $R = \Pi(L, \sigma_x)$ be the orbit algebra defined by an efficient tubular shift $\sigma_x$ at $x$. Then the prime element $\pi_x$ associated with $x$ is central in $R$.

Proof. Write $\sigma = \sigma_x$ and $\pi = \pi_x$. Since for all homogeneous elements $r \in R$ of degree $n \geq 0$ we have the commutative universal diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\pi} & \sigma L & \xrightarrow{\sigma x^{e(x)}} & L & \xrightarrow{r_x} & 0 \\
& & \downarrow r & & \downarrow \sigma r & & \downarrow r_x & & \\
0 & \longrightarrow & \sigma^n L & \xrightarrow{\sigma^n \pi} & \sigma^{n+1} L & \xrightarrow{\sigma^{n+1} x^{e(x)}} & L & \xrightarrow{r_{n+1}} & 0,
\end{array}
\]
the element $\pi$ is central. \(\square\)

The non-simple bimodule case.

1.7.2. Let $X$ be a homogeneous exceptional curve and $M = F_M G$ be the underlying tame bimodule which we assume to be of type $(2, 2)$ and to be non-simple. We identify $F = G$. Let $\pi_x, \pi_y$ be an $F$-basis of $M$ as
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1.7.1. Let $\pi_x$ correspond to a unirational point $x$ (which exists by 0.6.2). That is, $\pi_x$ is defined as the kernel of the universal exact sequence

$$0 \longrightarrow L \xrightarrow{\pi_x} \sigma_x L \longrightarrow S_x \longrightarrow 0$$

where we identified $M$ with $\text{Hom}(L, \sigma_x L)$.

Let $\sigma = \sigma_x$ be the tubular shift associated to the unirational point $x$, and let $R = \Pi(L, \sigma)$ be the corresponding orbit algebra. Then $\pi_x$ and $\pi_y$ are elements in $R_1 = M$, and (at least) $\pi_x$ is a prime element in $R$. (Note that $\pi_y$ can be non-prime.) By Lemma 1.7.1 the element $\pi_x$ is central.

By [87] there are $\alpha, \delta : F \longrightarrow F$ such that for all $f \in F$ the formula

$$\pi_y f = \delta(f)\pi_x + \alpha(f)\pi_y$$

holds, where $\alpha$ is a $k$-automorphism of $F$ and $\delta$ is an $(\alpha, 1)$-derivation on $F$. Since $\dim R_n = n + 1$, it is easy to see that the $n + 1$ elements $\pi^n_x, \pi^{n-1}_x\pi_y, \pi^{n-2}_x\pi^2_y, \ldots, \pi_x\pi^{n-1}_y, \pi^n_y$ form a $R_0$-basis of $R_n$ for each natural number $n$.

Denote by $F[X; Y, \alpha, \delta]$ the skew polynomial ring in two variables, where every element is expressible uniquely in the form $\sum_{i,j} f_{ij}X^iY^j$ with $f_{ij} \in F$ (that is, as left polynomial) and such that $X$ is central and for all $f \in F$ we have

$$Yf = \delta(f)X + \alpha(f)Y.$$ 

Since $\alpha$ is bijective each element can also be expressed uniquely as right polynomial. Moreover, this ring is graded by total degree (that is, $\deg(X) = 1 = \deg(Y)$). Then we get

**Proposition 1.7.3.** As graded algebras $\Pi(L, \sigma_x) \simeq F[X; Y, \alpha, \delta]$. \(\square\)

Therefore this case is also referred to as the skew polynomial case.

**Remark 1.7.4.** (1) See [87, 21] for an affine version of this. Therefore the preceding result is not surprising. One should expect that for simple bimodules (of any numerical type) one gets in a similar way graded analogues of the rings considered in [28, 22] and [17, 5.3].

(2) The function fields in the simple bimodule cases are well known, see [88]. From the preceding proposition it follows again that $k(X) \simeq F(T, \alpha, \delta)$, the quotient division ring of the skew polynomial ring $F[T, \alpha, \delta]$. (The variable $T$ is obtained as $T = YX^{-1}$.)

(3) The factoriality of the skew polynomial algebras in 1.7.3 also follows from results by Chatters and Jordan [12].

**Lemma 1.7.5.** Let $R = F[X; Y, \alpha, \delta]$ where $X$ is central and $X$ and $Y$ having degree one. The homogeneous prime ideals of height one different from $RX$ are in one-to-one correspondence with the non-zero prime ideals in $F[Z, \alpha, \delta]$. 


Proof. Let $R_X$ be the localization with respect to the central multiplicative set given by the powers $X^n$ ($n \geq 0$). (See also Section 2.2.) By [35, 9.22] the homogeneous prime ideals disjoint from this set correspond to the homogeneous prime ideals in $R_X$. Restriction to the zero component gives the skew polynomial ring $F[Z, \alpha, \delta]$ in one variable, where $Z = X^{-1}$. The prime ideals in this ring are in one-to-one correspondence with the homogeneous prime ideals in $R_X$ since there is a central unit of degree one in $R_X$. □

Lemma 1.7.6. In Proposition 1.7.3 one can assume either $\alpha = 1_F$ or $\delta = 0$. If char $k = 0$ then one can assume $\delta = 0$.

Proof. The first assertion follows by adopting [43, 1.1.21] to the case of two variables. If char $k = 0$ and $\Pi(L, \sigma_x) \simeq F[X; Y, 1_F, \delta]$, then one can adopt [35, 9.23] to show that $\delta$ is an inner derivation, and hence as graded algebras $\Pi(L, \sigma_x) \simeq F[X; Y, 1_F, 0]$. (If $\delta$ is not inner, then by [35, 9.23] the ring $F[Z, 1_F, \delta]$ is a simple ring. Then by 1.7.5, $X$ consists of precisely one point. But since $k$ is infinite there are infinitely many points by [87, Thm. 3].)

We give an alternative argument: The zero component of the localization $R_X$ is finitely generated over its centre ([$17$, 5.2]); on the other hand it coincides with $F[Z, 1_F, \delta]$, and in case $\delta$ is not inner this is not finitely generated over its centre by Amitsur's theorem (compare [43, 1.1.32]).) □

If $\delta = 0$ then the non-simple bimodule case is also called the twisted polynomial case, if $\alpha = 1$ it is also called the differential polynomial case.

From now on we assume $\delta = 0$, so that $R = F[X; Y, \alpha]$ is a graded twisted polynomial algebra. Then also $\pi_Y$ is normal and hence prime. Let $\sigma_y$ be the corresponding tubular shift. Modulo inner automorphisms $\alpha$ has finite order $r$. There is some $u \in F^*$ such that $\alpha^r(f) = u^{-1}fu$ for all $f \in F$. We have $M = F \oplus F$ with $f \cdot (x, y) = (fx, fy)$ and $(x, y) \cdot f = (xf, y\alpha(f))$ for all $f$, $x$, $y \in F$. We write also $M = M(F, \alpha)$.

Denote by $\text{Fix}(\alpha)$ the subfield of all $f \in F$ such that $\alpha(f) = f$. With $u$ as above, one can assume that $u \in \text{Fix}(\alpha)^*$. Let $K = Z \cap \text{Fix}(\alpha)$.

Lemma 1.7.7. The centre of $R = F[X; Y, \alpha]$ is given by $K[X, uY^r]$. The homogeneous prime elements in $R$ are (up to multiplication with a unit) $X$, $Y$ and the homogeneous prime elements in $K[X, uY^r]$, which are polynomials in $X^r$ and $uY^r$ with coefficients lying in $K$.

Proof. This follows by reducing to [43, 1.1.22] as in the proof of 1.7.5. □

(Note that the centre can be also determined in the graded differential polynomial case $R = F[X; Y, 1_F, \delta]$, compare [43, 1.1.32].)
It follows in particular that (up to multiplication with a unit) every homogeneous prime element in $R$ except $Y$ is central. ($Y$ itself is central (up to a unit) only in case $r = 1$.) Note that for example the central elements of the form $aX^r + bY^r$ with $a, b \in K^*$ are prime in $R$.

There is a version for the function field:

1.7.8. The centre of the function field $F(T, \alpha)$ is given by $K(uT^r)$, with the notations as above. The dimension of $F(T, \alpha)$ over its centre is given by $r^2m^2$ with $m^2 = [F : Z(F)]$. (We call the number $s(X) = rm$ the (global) skewness.)

**Lemma 1.7.9.** Let $z$ be a point different from the points $x$ and $y$ which correspond to the prime elements $X$ and $Y$, respectively. Then $d(z) = e(z) \cdot f(z)$ is a multiple of $r$.

**Proof.** The corresponding prime element $\pi_z$ is a product of $e = e(z)$ many irreducible elements, each of degree $f = f(z)$. Now $\pi_z$ is a homogeneous polynomial in the variables $X^r$ and $uY^r$ of degree $d = ef$. □

For an element $f \in F$ and each integer $i \geq 0$ let $N_i(f) = \alpha_i^{-1}(f) \cdots \alpha(f)f$. We call $N(f) = N_r(f)$ the norm of $f$.

**Lemma 1.7.10.** With the notations as above, assume that there is an element $b \in F$ such that $u^{-1} = N(b)$. Then the element $u^{-1}(X^r - uY^r)$ is a product of $r$ irreducible elements in $R$. Accordingly, the multiplicity of the corresponding rational point $z$ is given by $e(z) = r$, and the endomorphism ring of the corresponding simple object $S_z$ is given by the skew field of those elements $f \in F$ such that $\alpha(f) = b^{-1}fb$.

**Proof.** Follows from [43, 1.3.12] and 1.6.5. □

Explicit examples over the real numbers with complete lists of prime elements are given in 5.6.1.

1.7.11 (Arbitrarily large multiplicities). If $F$ is commutative then one can assume that $u = 1$. If furthermore $F/k$ is a cyclic Galois extension of degree $r$ with Galois group generated by $\alpha$ then the preceding lemma can be applied. In this way it is possible to construct examples of exceptional curves having points (even rational ones) with arbitrarily large multiplicities.

The dimension of $k(X)$ over its centre is always a perfect square. This dimension can also be arbitrarily large which follows from the same example. In the present case $k(X) = F(T, \alpha)$ has dimension $r^2$ over its centre $k(T^r)$.

It will follow from 2.2.13 that in this case $r$ is the maximal multiplicity.
The quaternion case. Let $k$ be a field of characteristic different from two. Let $a, b \in k^*$ and let $F = \left( \frac{a}{b} \right)$ be an algebra of quaternions over $k$, that is, a $k$-algebra on generators $i$ and $j$ subject to the relations $ji = -ij$, $i^2 = a$, $j^2 = b$.

We assume that $F$ is a skew field. Equivalently, the norm form of pure quaternions $-aX^2 - bY^2 + abZ^2$ is anisotropic over $k$. Let $M$ be the bimodule $kF_F$.

We have shown in [53] that the small preprojective algebra is given by $\Pi(L, \tau^{-}) \simeq k[X, Y, Z]/(-aX^2 - bY^2 + abZ^2)$.

We will see later that $\tau^{-}$ is the only efficient automorphism in this case.

Note that the factoriality of this algebra was already known from a theorem of P. Samuel [97], we refer to [32, Prop. 11.5].

It is interesting that the bimodule $kF_F$ given by noncommutative data gives a commutative orbit algebra. The next example shows the converse behaviour. A reason for this will be explained in Section 4.3.

The square roots case. The following example is based on calculations by D. Baer [5, 1.3.6]. In order to get the following result we need an additional argument.

Proposition 1.7.12. Let $a, c \in k$ and $K$ the field $k(\sqrt{a}, \sqrt{c})$ such that $[K:k] = 4$. Let $M$ be the $k$-$K$-bimodule $K$ and let $X$ be the associated homogeneous exceptional curve. Then there is a unirational point $x \in X$ such that $\Pi(L, \sigma_x) \simeq k[X, Y, Z]/(XY - YX, XZ - ZX, YZ + ZY, Z^2 + aY^2 - cX^2)$, where each variable is of degree one. In particular, the algebra on the right hand side is graded factorial.

Proof. In [5, 1.3.6] the category of preprojective modules of rank one over the associated bimodule algebra, which is equivalent to the category $L_+$ as defined in 1.4.1, is determined explicitly. It is easy to see that the morphism $Y$ between two successive preprojectives of rank one described in [5] defines a universal extension with simple cokernel given as representation $S_x : k^2 \otimes K \rightarrow (1, \sqrt{a})K$ (with endomorphism ring $k(\sqrt{a})$), and that the degree shift given in [5] coincides on the category $L_+$ with the tubular shift $\sigma_x$. Now the assertion follows from the relations determined in [5] (where we changed the roles of the variables $X$ and $Y$).

For further properties of this example see 4.3.7, 3.2.14 (2) and 5.7.2. Explicit examples are given by $k = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ or $K = \mathbb{Q}(\sqrt{2}, -\sqrt{3})$ (the latter case occurring in 8.3.2).
CHAPTER 2

Global and local structure of the sheaf category

As a consequence of a version of Serre’s theorem by M. Artin and J. J. Zhang the graded factorial orbit algebras considered in the previous chapter are projective coordinate algebras for the homogeneous exceptional curves. It follows from the graded factoriality that homogeneous prime ideals of height one are localizable. We also consider the localization with respect to the multiplicative set given by the powers of a prime element. We describe properties of these localizations and derive important relations between the multiplicity function \( x \mapsto e(x) \) and the dimension of the function field \( k(X) \) over its centre.

2.1. Serre’s theorem

Let \( X \) be a homogeneous exceptional curve with hereditary category \( \mathcal{H} \). Let \( E \) be a non-zero object in \( \mathcal{H}_+ \), and let \( \varphi \) be some automorphism of \( \mathcal{H} \). Recall that \( \varphi \) is positive if \( \deg(\varphi L) > 0 \). In \([2]\) (see also \([103]\)) the pair \((E, \varphi)\) is called ample if the following holds:

1. For each object \( X \in \mathcal{H} \) there is an epimorphism \( \bigoplus_{i=1}^n \varphi^{-\alpha_i}(E) \twoheadrightarrow X \) with integers \( \alpha_i > 0 \).
2. Each epimorphism \( X \twoheadrightarrow Y \) induces an epimorphism

\[
\text{Hom}(E, \varphi^nX) \twoheadrightarrow \text{Hom}(E, \varphi^nY)
\]

for \( n \gg 0 \).

**Lemma 2.1.1.** If \( \varphi \) is positive then the pair \((E, \varphi)\) is ample.

**Proof.** From the special structure of \( \mathcal{H} \), the first property for ampleness is true for \( E = L \) and \( E = \mathcal{T} \) and then follows easily for arbitrary \( E \). The second property for ampleness follows from 0.4.6 with Serre duality. \( \square \)

The following theorem is a special case of Serre’s theorem for non-commutative projective schemes by M. Artin and J. J. Zhang \([2]\).

**Proposition 2.1.2 (Serre’s theorem).** Let \( X \) be a homogeneous exceptional curve. Let \( E \) be a non-zero vector bundle and \( \varphi \) be a positive automorphism of \( \mathcal{H} \). Let \( R = \Pi(E, \varphi) = \bigoplus_{n \geq 0} \text{Hom}_\mathcal{H}(E, \varphi^nE) \) be the orbit algebra

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defined to the pair \((E, \phi)\). Then the section functors
\[
\Gamma : \mathcal{H} \rightarrow \text{Mod}^\mathbb{Z}(R), \quad F \mapsto \bigoplus_{n \in \mathbb{Z}} \text{Hom}(E, \phi^n(F))
\]
and
\[
\Gamma_+ : \mathcal{H} \rightarrow \text{mod}^\mathbb{Z}_+(R), \quad F \mapsto \bigoplus_{n \geq 0} \text{Hom}(E, \phi^n(F))
\]
induce equivalences
\[
\mathcal{H} \simeq \frac{\text{mod}^\mathbb{Z}(R)}{\text{mod}_0^\mathbb{Z}(R)} \simeq \frac{\text{mod}^\mathbb{Z}_+(R)}{\text{mod}_0^\mathbb{Z}_+(R)}.
\]

**Corollary 2.1.3.** Each homogeneous exceptional curve admits a (non-commutative) graded factorial domain as projective coordinate algebra. □

**Corollary 2.1.4.** Let \(X\) be an exceptional curve. Then the function field \(k(X)\) is the quotient division ring of degree zero fractions of a graded factorial domain.

**Proof.** Note that the function field of an exceptional curve coincides with the function field of the underlying homogeneous curve. If \(X\) is homogeneous and \(R = \Pi(L, \sigma)\) (with \(\sigma\) efficient) then by \([6]\) (compare also \([86, IV.4.1, Step 4]\)) \(k(X)\) coincides with the degree zero part of the graded quotient division ring \(\text{Quot}^\mathbb{Z}(R)\) of \(R\). □

2.1.5 (The sheafification functor). Let \(R = \Pi(E, \varphi)\) be with \(\varphi\) positive. (Of course, we have in mind \(E = L\) and \(\varphi = \sigma\) being efficient.) Denote by \(T : \text{mod}^\mathbb{Z}(R) \rightarrow \text{mod}^\mathbb{Z}(R)/\text{mod}_0^\mathbb{Z}(R) \simeq \text{mod}^\mathbb{Z}_+(R)/\text{mod}_0^\mathbb{Z}_+(R)\) the canonical quotient functor. By the proposition \(TT_+\) is an equivalence. Denote by \(\phi : \frac{\text{mod}^\mathbb{Z}(R)}{\text{mod}_0^\mathbb{Z}(R)} \rightarrow \mathcal{H}\) some quasi-inverse and define \(\tilde{\cdot} : \text{mod}^\mathbb{Z}(R) \rightarrow \mathcal{H}\) (and also its restriction to \(\text{mod}^\mathbb{Z}_+(R)\)) by \(\tilde{\cdot} = \phi T\). We may assume that \(\widetilde{R(n)} = \varphi^n E\) (usually \(= L(n)\)) for all \(n \in \mathbb{Z}\). This gives an exact and dense functor with kernel \(\text{mod}_0^\mathbb{Z}(R)\) and such that \(\tilde{\cdot} \circ \Gamma_+ \simeq 1\).

Denote also by \(T\) the quotient functor \(T : \text{Mod}^\mathbb{Z}(R) \rightarrow \frac{\text{Mod}^\mathbb{Z}(R)}{\text{Mod}_0^\mathbb{Z}(R)}\). Then \(T\) admits a right adjoint \(S\) (the section functor) which is fully faithful and \(TS = 1\) holds (compare \([83, 4.4]\)). Denote by \(S_+ : \frac{\text{mod}^\mathbb{Z}_+(R)}{\text{mod}_0^\mathbb{Z}_+(R)} \rightarrow \text{mod}^\mathbb{Z}_+(R)\) the functor given by \(S_+ T(M) = ST(M)_{\geq 0}\) (the non-negative part of \(ST(M)\)) for each \(M \in \text{mod}^\mathbb{Z}_+(R)\). This is well-defined, since by \([2, 4.5\ S5-S7]\) \(T(M) \simeq TT(\cdot)\) for some \(F \in \mathcal{H}\), and \(ST(M)_{\geq 0} \simeq ST(\cdot)_{\geq 0} \simeq \Gamma(\cdot)_{\geq 0} = \Gamma_+(\cdot)\) is finitely generated. Since for all \(M\) and \(N\) in \(\text{mod}^\mathbb{Z}_+(R)\) obviously
Hom(M, ST(N) ≥ 0) = Hom(M, ST(N)), it follows directly by the adjointness of T and S that also (T, S+) is an adjoint pair, that is, \( \text{mod}^{Z+}(R) \) is a localizing subcategory of \( \text{mod}^{Z+}(R) \). By the preceding argument, \( S^+T\Gamma_+ \cong \Gamma_+ \), and thus \( \Gamma_+ \) is fully faithful. Since there is a natural transformation \( 1 \rightarrow S^+T \cong \Gamma_+ \circ \sim \), it follows easily that \( \Gamma_+ \) is right adjoint to \( \sim \). \( \square \)

2.1.6 (Degree shift). We keep the notation from the preceding number. Denote by \( D \) the degree shift \( X \mapsto X(1) \) on \( \text{Mod}^{Z}(R) \). Then \( D\Gamma = \Gamma\varphi \).

Denote by \( D_+ \) the functor on \( \text{mod}^{Z+}(R) \) given by \( D_+(M) = D(M)_{\geq 0} \). Then similarly \( D_+\Gamma_+ = \Gamma_+\varphi \). It follows that via the equivalences \( TT\Gamma_+ \) and \( \phi \) the automorphism \( \varphi \) (defining the grading of \( R \)) corresponds to the automorphism on \( \frac{\text{mod}^{Z+}(R)}{\text{mod}^{Z+}(R)} \) which is induced by \( D_+ \).

2.1.7 (Section modules/Cohen-Macaulay modules). For simplicity, we return to our standard situation, that is, \( R = \Pi(L, \sigma) \) with \( \sigma \) being efficient. Then \( R_0 = \text{End}(L) \) is a skew field and is up to shift the only simple graded \( R \)-module. Denote by \( \text{sect}^{Z+}(R) \) the full subcategory of \( \text{mod}^{Z+}(R) \) formed by those \( M \) such that

\[
\text{Hom}_R(R_0(-n), M) = 0 = \text{Ext}^1_R(R_0(-n), M)
\]

for all \( n \geq 0 \). We have \( M \in \text{sect}^{Z+}(R) \) if and only if there is \( F \in \mathcal{H} \) such that \( M \cong \Gamma_+(F) \). In fact, from \( M \cong \Gamma_+(F) \) we get \( \tilde{M} \cong F \). Moreover, it can be deduced from [2, 3.14] that \( M \in \text{sect}^{Z+}(R) \) if and only if \( M \cong \Gamma_+(\tilde{M}) \).

Thus, the section functor \( \Gamma_+ \) induces an equivalence \( \mathcal{H} \cong \text{sect}^{Z+}(R) \).

Similarly, denote by \( \text{CM}^{Z}(R) \) the full subcategory of \( \text{mod}^{Z}(R) \) formed by those modules \( M \) such that

\[
\text{Hom}_R(R_0(n), M) = 0 = \text{Ext}^1_R(R_0(n), M)
\]

for all \( n \in \mathbb{Z} \), that is, \( \text{CM}^{Z}(R) = \text{mod}^{Z}(R)^+ \). The objects in \( \text{CM}^{Z}(R) \) are called graded maximal Cohen-Macaulay modules. \( \Gamma \) induces an equivalence \( \mathcal{H}_+ \cong \text{CM}^{Z}(R) \). For this, because of [2, 3.14] it is sufficient to show, that \( \Gamma(F) \) is finitely generated for each \( F \in \mathcal{H}_+ \). Since by [6, 2.4] \( F \) is a subobject of a finite direct sum of shifts of \( L \) (and \( L \), if \( L \) is a line bundle, which then is a subobject of \( L(1) \)), this follows by left exactness of \( \Gamma \) and noetherianness of \( R \). \( \square \)

2.1.8 (A Koszul complex). Let \( R = \Pi(L, \sigma) \) with \( \sigma \) being an efficient automorphism. \( R \) has the Cohen-Macaulay property in the sense of regular sequences, since any two non-associated prime elements \( \pi_1 \) and \( \pi_2 \) define a regular sequence. For the (two-sided) ideal \( I = \pi_1 R + \pi_2 R \) the factor \( R/I \) has Krull dimension zero, and hence is of finite length [35], and therefore finite dimensional.
Let $d_1$ and $d_2$ be the degrees of $\pi_1$ and $\pi_2$, respectively, and assume additionally that $\pi_1 \pi_2 = \pi_2 \pi_1$. Then this regular sequence defines a projective resolution of the graded right $R$-module $R/I$:

$$
0 \longrightarrow R(-d_1 - d_2) \longrightarrow R(-d_1) \oplus R(-d_2) \longrightarrow R \longrightarrow R/I.
$$

By sheafification this leads to the exact sequence

$$
0 \longrightarrow L(-d_1 - d_2) \longrightarrow L(-d_1) \oplus L(-d_2) \longrightarrow L \longrightarrow 0.
$$

\[\square\]

### 2.2. Localization at prime ideals

Let $R$ be a graded ring and $S \subset R$ be a multiplicative set consisting of homogeneous elements. Following [35] we call $S$ right Ore, if $rS \cap sR$ is non-empty for all $r \in R$ and $s \in S$. It is called a right reversible set, if for all $r \in R$ and $s \in S$ such that $sr = 0$ there is $s' \in S$ such that $rs' = 0$. (For both conditions it is sufficient to consider homogeneous elements $r$, see [82, I.6.1].) $S$ is called a right denominator set if it is right Ore and right reversible. The left hand versions are defined similarly. $S$ is called denominator set, if it is a left and right denominator set.

Let $X$ be a homogeneous exceptional curve and $R = \Pi(L, \sigma)$ with $\sigma$ being efficient. Since $R$ is a graded domain, reversibility holds automatically for multiplicative sets. Moreover, each multiplicative set consisting of normal elements is a denominator set. For example, if $f \neq 0$ is normal, then $\{f^n \mid n \geq 0\}$ is a denominator set. Denote by $R_f = R[f^{-1}]$ the corresponding ring of fractions.

Let $P = R\pi$ be a prime ideal of height one and let $x \in X$ be the corresponding point. We define the following multiplicative subsets of $R$ consisting of homogeneous elements: $N(P)$ contains all the normal elements not lying in $P$. The set $Y$ consists of those non-zero homogeneous elements, where the induced map between line bundles has cokernel contained in $\bigsqcup_{y \neq x} U_y$, that is, is not supported by $x$. Let $Y'$ be the set of all homogeneous elements $s$ such that the fibre map $s_x$ is an isomorphism. Finally, denote by $C(P)$ the set of homogeneous elements in $R$ whose classes are (left and right) regular (that is, non-zero divisors) in $R/P$.

**Lemma 2.2.1.** We have $N(P) \subset Y = Y' = C(P)$. All these multiplicative sets are denominator sets.
Proof. It is easy to see that all these sets are multiplicative and that \( N(P) \subset Y \subset Y' \subset C(P) \).

\( C(P) \) consists of those (homogeneous) \( s \in R \) such that \( s_x \in M_x(D) \) is regular (\( D \) a skew field), hence invertible (compare the proof of Theorem 1.2.3). Using Lemma 1.3.3, the equality \( Y = Y' \) follows easily. Obviously, \( N(P) \) is a denominator set. That \( C(P) \) is a denominator set follows from a graded version of \([12, \text{Lemma 2.2}]\). In orbit case I and II we might use also 1.6.7 for the proof that \( Y' \) is a right Ore set: Let \( r \in R \) be homogeneous and \( s \in Y \). Without loss of generality let \( s \) be irreducible. There is some \( s_1 \) and a prime element \( \pi_1 \) such that \( ss_1 = \pi_1 \), where \( R\pi_1 \neq R\pi \). Then

\[ r\pi_1 = \pi_1 r' = s(s_1 r') \]

for some \( r' \), since \( \pi_1 \) is normal. The left hand version is similar. □

Lemma 2.2.2. Assume orbit case I or II. Then the graded rings of fractions \( R_{N(P)} \) and \( R_{C(P)} \) are isomorphic via the natural morphism.

Proof. (Compare also \([94, \text{3.1.7}]\).) Let \( rs^{-1} \in R_{C(P)} \). Compose \( s = u_1 \ldots u_t \) into irreducibles. Then all \( u_i \in C(P) \). By 1.6.7 there are \( s_i \) and primes \( \pi_i \) such that \( \pi_i = u_is_i \). Then

\[ rs^{-1} = ((rs_1)\pi_1^{-1})(s_2\pi_2^{-1})\ldots(s_1\pi_1^{-1}) \in R_{N(P)}. \]

Hence the natural morphism is surjective. Injectivity is proved along the same lines: If \( rt^{-1} = 0 \) in \( R_{C(P)} \), then there is some \( s \in C(P) \) with \( rs = 0 \). Decomposing \( s \) as above into irreducibles and representing irreducibles as divisors of primes as above we see that there is some \( s' \in N(P) \) such that \( rs' = 0 \), and hence \( rt^{-1} = 0 \) also in \( R_{N(P)} \). □

Problem 2.2.3. Assume orbit case III. Does \( R_{N(P)} \simeq R_{C(P)} \) also hold in this case? Is each irreducible element a (left and right) divisor of a normal element?

We denote the graded ring of fractions \( R_{C(P)} \) by \( R_P \), its zero component by \( R_0 \). If \( P \) is corresponding to the point \( x \), we also write \( R_x \) and \( R_0 \), respectively. We have

\[ R_{N(P)} = \bigcup_{\substack{f \notin P \\text{normal}}} R_f, \]

where \( R_f \) denotes the ring of fractions of the form \( rf^{-n} \), with \( r \in R \) and \( n \geq 0 \). Denote by \( \text{HOM}(F, G) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(F, G(n)) \) and \( \text{END}(F) = \text{HOM}(F, F) \).

Denote by \( \mathcal{H}_x \) the quotient category of \( \mathcal{H} \) modulo the Serre subcategory spanned by \( \bigcup_{y \neq x} U_y \). Denote by \( q_x : \mathcal{H} \to \mathcal{H}_x, F \mapsto F_x \) the canonical functor. We refer to \([6]\) for more details about this category.
By a similar argument, each line bundle in $H^8_{x7}$ of such an element follows from $[M_{\text{zero component}}, R_{\text{element in a principal ideal domain}}].$

It is easy to check that this is well-defined and gives an isomorphism of graded rings $\rho : R_{P} \longrightarrow \text{END}_{H_{x}}(L_{x}).$

PROPOSITION 2.2.4. Let $L_{x}$ be the image of $L$ in $H_{x}.$ Then
\[ R_{P} \simeq \text{END}_{H_{x}}(L_{x}). \]

PROOF. For homogeneous as $a^{-1} \in R_{P},$ where $a$ (resp. $s$) is of degree $m$ (resp. $n$) in $R,$ let $\rho(a^{-1}) = \sigma^{-n}(a) \circ (\sigma^{-n}(s))^{-1} \in \text{Hom}_{H_{x}}(L_{x}, L_{x}(m-n)).$ It is easy to check that this is well-defined and gives an isomorphism of graded rings $\rho : R_{P} \longrightarrow \text{END}_{H_{x}}(L_{x}).$

COROLLARY 2.2.5. Let $x, y \in X$ be points so that there is an automorphism $\varphi \in \text{Aut} (X)$ with $\varphi (x) = y.$ Then $R_{x}^{0} \simeq R_{y}^{0}.$

PROOF. The autoequivalence $\varphi$ of $H$ induces an equivalence $H_{x} \simeq H_{y}$ mapping $L_{x}$ to $L_{y}$.

Recall that a (not necessarily commutative) ring is called a principal ideal domain if it is a domain and if every left ideal and every right ideal is generated by a single element.

PROPOSITION 2.2.6. $H_{x} \simeq \text{mod}^{Z}(R_{P}) \simeq \text{mod}(R_{P}^{0}).$ In particular, $R_{P}^{0}$ is a principal ideal domain.

PROOF. Note that $R_{P}$ contains a unit $u$ of degree one (take any non-zero element in $R_{1}$ with cokernel whose support is disjoint from $x;$ the existence of such an element follows from [87, 3.6]). Therefore, the restriction to the zero component, $M \mapsto M_{0},$ induces an equivalence $\text{mod}^{Z}(R_{P}) \simeq \text{mod}(R_{P}^{0}).$

By a similar argument, each line bundle in $H$ becomes isomorphic to $L_{x}$ in $H_{x}.$ Moreover, $L_{x}$ is a projective generator of $H_{x}.$ It follows that $H_{x} \simeq \text{mod}(\text{End}_{H_{x}}(L_{x})), \text{induced}$ by the functor $\text{Hom}_{H_{x}}(L_{x}, -)$ (compare also [7, II.1.3]). By the result before, $\text{End}_{H_{x}}(L_{x}) \simeq R_{P}^{0}.$

As before, let $\Gamma (F) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(L, F(n)) \in \text{Mod}^{Z}(R),$ and let $q_{P} : \text{Mod}^{Z}(R) \longrightarrow \text{Mod}^{Z}(R_{P})$ be the canonical functor. It is easy to see that each element of $\text{Ext}^{1}(L, F(n))$ can be annihilated by an element in $R$ of sufficiently high degree (by 0.4.6), and that each morphism $L \longrightarrow S_{y}$ is annihilated by the element of $R$ which is given by the kernel. It follows that the composition $q_{P} \circ \Gamma$ induces an exact functor $H \longrightarrow \text{mod}^{Z}(R_{P})$ such that for all $F \in \bigoplus_{y \neq x} H_{y}$ we have $q_{P} \circ \Gamma (F) = 0.$ Therefore we get the following diagram of functors:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{q_{P} \circ \Gamma} & \text{mod}^{Z}(R_{P}) \\
H_{x} & \xrightarrow{\phi} & \text{mod}(R_{P}^{0})
\end{array}
\]

$\phi$ is the unique (exact) functor defined on the quotient category $H_{x}$ such that $\phi \circ q_{P} = q_{P} \circ \Gamma,$ and moreover, $\phi (F) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(L, F(n))_{P}$ which is naturally isomorphic to $\bigoplus_{n \in \mathbb{Z}} \text{Hom}(L_{x}, F_{x}(n))$ (similar to Proposition 2.2.4).
Each non-zero subobject of \( L_x \) is isomorphic to \( L_x(-n) \) for some \( n \geq 0 \), hence \( R_P^0 \) is a right principal ideal domain. Since \( R_P^0 \) is noetherian it follows from [14] that \( R_P^0 \) is also a left principal ideal domain. \( \square \)

**Corollary 2.2.7** (The structure of tubes). \( \mathcal{U}_x \simeq \text{mod}_0(R_P^0) \), the modules of finite length over the principal ideal domain \( R_P^0 \). \( \square \)

For a similar statement with a complete local domain occurring compare [88, Thm. 4.2] and 2.2.16 below.

**Proposition 2.2.8.** Let \( P = R\pi \) be a homogeneous prime ideal of height one. There is a short exact sequence

\[
0 \longrightarrow R_P \longrightarrow R_P(d) \longrightarrow S_P^e \longrightarrow 0,
\]

where \( S_P \) is a simple graded right \( R_P \)-module and \( e \) the multiplicity of the corresponding point. In particular, \( R_P/P_P \) is a graded uniformly simple Artinian ring.

**Proof.** Let \( S \in \mathcal{H} \) be the simple object corresponding to \( P \). Then \( \phi \circ q_x(S) \) is a simple graded \( R_P \)-module, and applying the functor \( \phi \circ q_x \) to the \( S \)-universal sequence given by \( \pi \) induces the short exact sequence as in the assertion. \( \square \)

For the notion of a (not necessarily commutative) Dedekind domain we refer to [76].

**Corollary 2.2.9.** \( R_P \) is a graded Dedekind domain, that is, a noetherian hereditary domain such that each homogeneous, idempotent ideal equals 0 or \( R_P \). In \( R_P \) the only homogeneous prime ideals are 0 and \( P_P \), which is generated by a normal element. Moreover, \( P_P \) is the graded Jacobson radical of \( R_P \), and each graded right torsion module is unfaithful. Each non-zero homogeneous ideal is a power of \( P_P \).

**Proof.** Since \( R_P \) is a graded principal ideal domain, it follows that it is graded Dedekind. But there is also a direct argument: For each homogeneous element \( 0 \neq x \in R_P \) there is a natural number \( n = v(x) \) such that \( x \in P_P^n \), but \( x \notin P_P^{n+1} \). If \( I \neq 0, R_P \) is an idempotent ideal, choose a non-zero homogeneous element \( x \in I \) with \( v(x) \) minimal in order to get a contradiction.

Moreover, the annihilator of each graded simple right module is given by \( P_P \neq 0 \), which follows by a graded version of [76, 4.3.18]. The last statement follows by a graded version of [76, 5.2.9]. \( \square \)

**Theorem 2.2.10.** Let \( P = R\pi \) be a homogeneous prime ideal of height one and \( S \) be the associated simple sheaf. Let \( D \) be the endomorphism skew field \( \text{End}(S) \) and \( e \) the multiplicity of the corresponding point. Then \( P_P^0 \) is the unique non-zero prime ideal in \( R_P^0 \) and \( R_P^0/P_P^0 \simeq M_e(D) \).
Proof. In order to get the isomorphism apply the graded version of the Artin-Wedderburn theorem \cite{82} and restrict to the zero component. Note that the endomorphism ring and the simplicity of $S$ is preserved under the various functors we applied. It follows that $P^0_P$ is a maximal ideal in $R^0_P$, generated by the normal element $u^{-d} \pi 1^{-1}$ where $u \in R_P$ is a unit of degree one and $d$ the degree of $\pi$. By the principal ideal theorem \cite[4.1.11]{76} then $P^0_P$ is of height one.

Since $P_P$ is the graded Jacobson radical of $R_P$, the zero component $P^0_P$ is the Jacobson radical of $R^0_P$. The Jacobson radical is the intersection of all primitive ideals. It thus follows that $P^0_P$ is the unique maximal ideal in $R^0_P$, and hence it is the only non-zero prime ideal in $R^0_P$. \hfill \Box

Remark 2.2.11. It follows in particular that $R^0_P$ and $R_P$ are factorial (graded factorial, respectively).

Definition 2.2.12. Let $s(\mathcal{X})$ be the square root of the dimension of $k(\mathcal{X})$ over its centre,

\[ s(\mathcal{X}) = [k(\mathcal{X}) : Z(k(\mathcal{X}))]^{1/2}. \]

We call it the (global) skewness of $\mathcal{X}$. The curve $\mathcal{X}$ is commutative if and only if $s(\mathcal{X}) = 1$. For any point $x \in \mathcal{X}$ we call the square root $e^*(x)$ of the dimension of $\text{End}(S_x)$ over its centre the comultiplicity of the point $x$. (These numbers are just the PI degrees of the respective skew fields.) The results below indicate that the multiplicity $e(x)$ plays the role of a “local skewness”.

Corollary 2.2.13 (Upper bound for the multiplicities). The multiplicities are bounded by the skewness $s(\mathcal{X})$.

More precisely, with the comultiplicity $e^*(x)$ of a point $x$ we have

\[ e(x) \cdot e^*(x) \leq s(\mathcal{X}). \]

Proof. This is now a direct consequence of general results on polynomial identities in the context of the Amitsur-Levitzki theorem, see \cite[13.3]{76}. (The idea of the proof is due to W. Crawley-Boevey \cite{20}.) Let $e = e(x)$, $e^* = e^*(x)$, $s = s(\mathcal{X})$, $D = \text{End}(S_x)$ and $P$ be the corresponding prime ideal of height one. The localization $R^0_P$ is a subring of $k(\mathcal{X})$, and $M_e(D)$ is a factor ring of $R^0_P$. Since $k(\mathcal{X})$ satisfies a (monic) polynomial identity of degree $2s$ (that is, $s$ is the PI degree of $k(\mathcal{X})$) this holds also for $M_e(D)$. But $2ee^*$ is the smallest degree of a polynomial identity for $M_e(D)$, hence $ee^* \leq s$. \hfill \Box

In particular:

Corollary 2.2.14. If the function field $k(\mathcal{X})$ is commutative then $\mathcal{X}$ is multiplicity free and the endomorphism rings of the simple objects in $\mathcal{H}$ are commutative. \hfill \Box
2.3. NONCOMMUTATIVITY AND THE MULTIPlicITIES

In 4.3.1 it is shown that conversely the multiplicity freeness implies the commutativity.

The following simple observation is worth noting.

**Proposition 2.2.15.** If \( e = 1 \), then \( R^0_P \) is local. If \( e > 1 \), then \( R^0_P \) is not even semiperfect. The same is true for \( R_P \) in a graded version.

**Proof.** Since \( R^0_P \) is a domain, 0 and 1 are the only idempotents in \( R^0_P \). On the other hand, for \( e > 1 \) there are non-trivial idempotents in \( R^0_P/P^0_P \cong M_e(D) \).

\[ \square \]

2.2.16 (Completion). With the notations of 2.2.10, let \( \widehat{R^0_P} \) be the \( P^0_P \)-adic completion of \( R^0_P \). Then its Jacobson radical is given by \( P^0_P \). By [61, 21.31+23.10] \( R^0_P/P^0_P \) is a semiperfect ring with \( \widehat{R^0_P}/P^0_P \cong M_e(D) \) and there is a complete local ring \( S_P \) such that \( \widehat{R^0_P} \cong M_e(S_P) \), and \( U_x \cong \text{mod}_0(\widehat{R^0_P}) \cong \text{mod}_0(S_P) \) holds. Note that \( S_P \) does not longer contain information about the multiplicity \( e \).

The following corollary indicates that there should be no redundancies in a decomposition of a prime element into irreducible factors.

**Corollary 2.2.17.** Let \( \pi = u_1 u_2 \) be a factorization of a prime element \( \pi \) into irreducible elements \( u_1 \) and \( u_2 \). Then \( Ru_1 \neq Ru_2 \) and \( u_1 R \neq u_2 R \).

**Proof.** \( u_1 R \) and \( u_2 R \) induce direct summands, hence idempotents in the full matrix ring, and their product is zero.

\[ \square \]

We have the following property which is well-known for commutative integrally closed noetherian domains [75, 11.5].

**Proposition 2.2.18.**

\[ R = \bigcap_{\lambda(P)=1} R_N(P). \]

**Proof.** If \( s^{-1} r \in \cap R_N(P) \), where \( s \) is normal, one has to show that \( r \in sR \). By factorizing \( s \) in prime elements this follows easily. Compare also [12, Theorem 2.3].

\[ \square \]

2.3. Noncommutativity and the multiplicities

As in the previous section let \( X \) be homogeneous and \( R = \Pi(L, \sigma) \) with \( \sigma \) efficient. Instead of localizing at a prime ideal, which means to “remove” all other points, we now exploit the localization with respect to elements whose cokernels are concentrated in a given point, which means to “remove” just this point. As application we get that the inequation in 2.2.13 is generically an equation.
Lemma 2.3.1. Let $x \in \mathbb{X}$ be a rational point. The set of non-zero homogeneous elements $s \in R$ such that $\text{coker}(s) \in \mathcal{U}_x$ is a denominator set.

Proof. The right Ore condition: If $s : L \rightarrow L(n)$ is in the defined set, and $r : L(m) \rightarrow L(n)$ is a homogeneous element of $R$ (up to shift), consider the inclusions $i$ and $j$ of the pullback (intersection) $L'$ of $s$ and $r$ into $L$ and $L(n)$, respectively. If $L'$ is in the “wrong” orbit of line bundles, compose them with a monomorphism $f : L(p) \rightarrow L'$ with cokernel in $\mathcal{U}_x$, which exists since $x$ is rational. Then $sif = rjf$, and $\text{coker}(jf) \in \mathcal{U}_x$.

For the left Ore condition we consider similarly a pushout diagram of $r$ and $s$. If the obtained object of rank 1 decomposes, project to the line bundle summand. If this line bundle is in the “wrong” orbit, apply again a suitable map with cokernel in $\mathcal{U}_x$. (Compare [6, Lemma 2.6].) \hfill \Box

Let $x$ be a rational point. We denote the localization of $R$ with respect to the denominator set of the preceding lemma by $R_{(x)}$, its degree zero component by $R^0_{(x)}$. In this way we get as affine rings similar principal ideal domains like in [17, §5]:

Proposition 2.3.2. Let $x$ be a rational point.
1. As graded rings, $R_{(x)} \simeq \text{END}_{\mathcal{H}/(U_x)}(L)$.
2. $\mathcal{H}/(U_x) \simeq \text{mod}^\mathbb{Z}(R_{(x)}) \simeq \text{mod}(R^0_{(x)})$.
3. $R^0_{(x)}$ is a principal ideal domain.

Proof. Like in the preceding section. We only remark that there is a unit of degree one in $R_{(x)}$. In fact, since $x$ is rational, $\pi$ can be written as $\pi = uv$ where $u$ is irreducible and of degree one (compare 1.6.5 and 1.6.6), and then $(u \cdot 1^{-1}) \cdot (v \cdot 1^{-1}) = 1 \cdot 1^{-1}$. \hfill \Box

Lemma 2.3.3. Let $x$ be a rational point, $y \neq x$ and $P_y$ the homogeneous prime ideal associated to $y$. Let $r$ and $s$ be homogeneous elements of $R$ such that the cokernel of $s \neq 0$ lies in $\mathcal{U}_x$. Then $rs \in P_y$ implies $r \in P_y$.

Proof. The kernel and the cokernel of the fibre map $s_y$ are on the one hand concentrated in $x$, on the other hand in $y$, hence they are zero and $s_y$ is an isomorphism. By the proof of Theorem 1.2.3, $rs \in P_y$ means $\sigma^m(r_y) \circ s_y = (rs)_y = 0$ (where $m$ is the degree of $s$), and thus $r_y = 0$ follows, which means $r \in P_y$. \hfill \Box

Theorem 2.3.4. Let $x$ be a rational point. For each point $y$ with $y \neq x$ denote by $\pi'_y$ a normal generator of $\pi_y R_{(x)} \cap R^0_{(x)}$. Then
1. We have $R^0_{(x)}/\pi'_y R^0_{(x)} \simeq M_{e(y)}(D_y)$, where $D_y = \text{End}(S_y)$.
2. Let $U \subset \mathbb{X} \setminus \{x\}$ be an infinite subset. Then $\bigcap_{y \in U} \pi'_y R^0_{(x)} = 0$.
3. There is a point $y \in U$ such that $e(y) \cdot e^*(y) = s(X)$. 

Proof. (1) This follows as in 2.2.10 with a version of 2.2.8.

(2) Using the preceding lemma we get \( \bigcap_{y \neq x} \pi_y R(x) = 0 \) from 1.3.6, and intersecting with the component of degree zero gives the result.

(3) By (1) and (2) there is an inclusion \( R^0(x) \subset \prod_{y \in U} M_{e(y)}(D_y) \). Since the PI degree of \( R^0(x) \) coincides with the PI degree of its quotient division ring ([1, Thm. 7]), which is \( k(X) \) (since there is a unit in \( R(x) \) of degree one), the assertion follows. \( \square \)

Corollary 2.3.5. Let \( X \) be an exceptional curve. Then the equality
\[ e(y) \cdot e^*(y) = s(X) \]
holds generically, that is, for all points \( y \in X \) except finitely many.

Proof. Let \( x \) be rational. The set \( \{ y \in X \mid y \neq x, e(y) \cdot e^*(y) \neq s(X) \} \) must be finite by part (3) of the preceding proposition. (Obviously, this holds also in the weighted situation.) \( \square \)

Remark 2.3.6. In general points \( x \) such that \( e(x) \cdot e^*(x) \neq s(X) \) may exist. For example, for the bimodule \( M = \mathbb{C} \oplus \mathbb{C} \) over the real numbers this inequality holds precisely for the points corresponding to the prime ideals generated by \( X \) and \( Y \), respectively, in the orbit algebra \( \mathbb{C}[X; Y, \pi] \). (Note that these two points are just the separation points on the boundary. See 5.1.1 (5) for more details.)

It is an interesting question which role those finitely many points play where inequality holds and whether there is a connection to the ghost group.

Corollary 2.3.7. Assume that there is an infinite subset \( U \) of points \( x \) whose corresponding simple sheaves \( S_x \) all have commutative endomorphism rings. Then there is a point \( x \in U \) such that \( e(x) = s(X) \). \( \square \)

Corollary 2.3.8. Let \( X \) be an exceptional curve over a finite field \( k \). Then there is a point \( x \in X \) such that \( e(x) = s(X) \). Actually, this holds for all except finitely many points.

Proof. Since each finite skew field is commutative (Wedderburn), we have \( e^*(x) = 1 \) for all \( x \in X \). Since \( X \) contains always infinitely many points, the result follows from the preceding results. \( \square \)

Corollary 2.3.9. Let \( k \) be a finite field. Then \( X \) is commutative if and only if \( X \) is multiplicity free.

Proof. Follows directly from 2.2.13 and 2.3.8. \( \square \)

We will show in 4.3.1 that the preceding corollary holds over any field. Hence our results show that the multiplicity function \( e \) measures noncommutativity, locally and globally.
Problem 2.3.10. Is it true that \( s(X) \) is always the maximum of \( e \)? Is this maximum taken on even by a rational point? Are there always infinitely many points (even rational points, if \( k \) is infinite) \( x \) such that \( \text{End}(S_x) \) is commutative? (For the class of examples in 1.7.11 the answer to all three questions is positive.)

Problem 2.3.11. Understand the role each single multiplicity \( e(x) \) (and comultiplicity \( e^*(x) \)) plays in terms of the function field. In particular: Is each \( e(x) \) and \( e^*(x) \) or their product a divisor of \( s(X) \)?

2.4. Localizing with respect to the powers of a prime element

In this section we describe affine parts of the curve \( X \) by localizing with respect to the powers of certain prime elements.

Lemma 2.4.1. Let \( x \in X \), and assume that \( f(x) \) is even in orbit case III. Let \( \pi_x \in R \) be the corresponding prime element and \( S_x \) the corresponding simple object.

1. Let \( s \in R \) be homogeneous such that \( s \neq 0 \) and \( \text{coker}(s) \in U_x \). Then there is factorization \( s = u_1 \ldots u_t \), where each \( u_i \in R \) is irreducible such that \( \text{coker}(u_i) \simeq S_x \). Moreover, each \( u_i \) is a (left and right) divisor of \( \pi_x \).

2. The multiplicative subset of all homogeneous \( s \in R \) such that \( s \neq 0 \) and \( \text{coker}(s) \in U_x \) is a denominator set.

3. Each fraction \( rs^{-1} \), with \( s \neq 0 \) homogeneous such that \( \text{coker}(s) \in U_x \), can be written as \( r'\pi_x^{-n} \) for some \( r' \in R \) and some \( n \geq 0 \).

Proof. (1) The factorization of \( s \) follows as in the proof of Proposition 1.6.6 considering a chain of projections from the cokernel of \( s \) decreasing the length in each step by one; the cokernel of each \( u_i \) is isomorphic to \( S_x \) (since in orbit case III we assume that \( f(x) \) is even). Each \( u_i \) is a divisor of \( \pi_x \) by Lemma 1.6.7.

(2) Follows as in the proof of Lemma 2.2.1.

(3) Follows by a similar argument given in the proof of Lemma 2.2.2. □

2.4.2 (Quasi-rational points). In orbit case III, ring theoretically rational points behave more complicated in some sense than points \( x \) with \( f(x) = 2 \), compare Proposition 1.6.6, or the preceding lemma.

We call a point \( x \in X \) quasi-rational if the prime element \( \pi_x \) factors into irreducibles of degree one and with simple cokernel. That is,

\[
x \text{ is quasi-rational } \iff \begin{cases} f(x) = 1 & \text{in orbit cases I and II}, \\ f(x) = 2 & \text{in orbit case III}. \end{cases}
\]

Since rational points always exist, trivially also quasi-rational points exist in orbit cases I and II. In orbit case III, a quasi-rational point exists if
and only if there is a non-zero map \( f \in \text{Hom}(L, L(1)) \) which cannot be decomposed into a product of maps from \( \text{Hom}(L, L) \) and \( \text{Hom}(L, L(1)) \), that is, \( f \) is 1-irreducible. The existence of such a map is open in general, but one should expect that this follows by a similar dimension argument like in [28, 2.4].

**Problem 2.4.3.** Do quasi-rational points always (in orbit case III) exist?

**Proposition 2.4.4.** Let \( \pi = \pi_\sigma \in R \) be a prime element corresponding to a quasi-rational point \( x \). Let \( R_\pi = R[\pi^{-1}] = \{ r\pi^{-n} \mid r \in R, n \geq 0 \} \).

1. As graded rings, \( R_\pi \simeq \text{END}_{\mathcal{H}/U}(L) \).
2. \( \mathcal{H}/U \simeq \text{mod}^Z(R_\pi) \simeq \text{mod}(R_\pi^0) \).
3. \( R_\pi^0 \) is a principal ideal domain.

**Proof.** (1) Note that each homogeneous fraction in \( \text{END}_{\mathcal{H}/U}(L) \) can be written as fraction \( rs^{-1} \) such that \( r, s \in R \) are homogeneous with \( s \neq 0 \) and \( \text{coker}(s) \in U \). With the preceding lemma the assertion follows as in Proposition 2.2.4.

(2) and (3) follow similarly as in the proof of Proposition 2.2.6. Note that there is a unit of degree one in \( R_\pi \). In fact, since \( x \) is quasi-rational, \( \pi \) can be written as \( \pi = uv \) where \( u \) is irreducible and of degree one (compare 1.6.5 and 1.6.6), and then \((u \cdot 1^{-1}) \cdot (v \cdot \pi^{-1}) = 1 \cdot 1^{-1}\).

2.5. Zariski topology and sheafification

As before, let \( X \) be homogeneous with hereditary category \( \mathcal{H} \) and orbit algebra \( R = \Pi(L, \sigma) \), where \( \sigma \) is efficient.

From the Artin-Zhang version of Serre’s theorem we deduced some formal “sheafification functor” \( \overset{\sim}{-} : \text{mod}^Z(R) \longrightarrow \mathcal{H} \). In this section we briefly sketch how we get such a functor in a more explicit manner.

We identify \( X \) with the set of height one homogeneous prime ideals in \( R \), hence \( X = \text{Proj}(R) \). If \( f \) is a normal element, denote by \( X_f \) the subset of all prime ideals \( P \) such that \( f \notin P \). The following lemma is easy to show.

**Lemma 2.5.1.** The system of sets \( X_f \) (\( f \neq 0 \) normal, non-unit) forms a basis for a topology such that \( X \) is connected and the proper closed sets are precisely the finite subsets of \( X \).

It is routine to define graded coherent and quasi-coherent sheaves over \( X \) and we get a sheafification functor \( \overset{\sim}{-} : \text{Mod}^Z(R) \longrightarrow \text{Qcoh}(X) \), \( M \mapsto \overset{\sim}{M} \), where \( \overset{\sim}{M}(X_f) = M_f \), the localization of \( M \) with respect to the powers of \( f \), where the structure sheaf \( \mathcal{O}_X \) is defined by \( \mathcal{O}_X(X_f) = R_f \). For each point \( x \in X \) the stalk is given by the localization \( R_{\mathcal{V}(P)} \), where \( P \) is the homogeneous prime ideal corresponding to \( x \). By 2.2.18 it follows that \( \mathcal{O}_X(X) = R \). For
the global section functor \( \Gamma = \Gamma(\mathcal{X}, -) : \text{Qcoh}(\mathcal{X}) \to \text{Mod}^\sim(R) \) one shows the following properties:

- \( \Gamma \) is right adjoint to \( \tilde{\cdot} \) (using the natural morphism \( \rho_M : M \to \tilde{M}(\mathcal{X}) \), assigning to \( m \in M \) all the fractions \( m^{-1} \in M_f = \tilde{M}(\mathcal{X}_f) \)).
- \( \tilde{\cdot} \circ \Gamma \simeq 1 \).

Moreover, there is the following version of Serre’s theorem.

**Proposition 2.5.2 (Serre’s theorem).** Sheafification \( \tilde{\cdot} : \text{mod}^\sim(R) \to \text{coh}(\mathcal{X}) \)

induces an equivalence

\[
\text{coh}(\mathcal{X}) \simeq \frac{\text{mod}^\sim(R)}{\text{mod}_0^\sim(R)}.
\]

**Proof.** We only show that \( \text{mod}_0^\sim(R) \) is the kernel of the sheafification functor. Denote by \( m \) the graded Jacobson radical of \( R \). We have \( \tilde{M} = 0 \) if and only if \( M_f = 0 \) for all normal elements \( f \in m \). Since each graded simple \( R \)-module is of the form \( (R/m)(n) \) for some \( n \in \mathbb{Z} \), we see that \( \tilde{M} = 0 \) if \( M \) is of finite length.

For \( M \in \text{mod}^\sim(R) \) there is a finite sequence \( 0 = M_0 \subset \cdots \subset M_s = M \) of submodules such that \( M_i/M_{i-1} \) is a fully faithful (and torsionfree) left \( R/P_i \)-module, where \( P_i \) is the corresponding affiliated prime ideal (see [35, 2.8]). If \( \tilde{M} = 0 \), then for all normal \( f \in m \) it follows that \( f^n \) is contained in the annihilator of \( M \) for some \( n \geq 0 \), hence \( f \in P_i \) and thus \( P_i = m \) for all \( i = 1, \ldots, s \). Therefore \( M \) is of finite length. \( \Box \)

**Corollary 2.5.3.** \( \mathcal{H} \) is equivalent to \( \text{coh}(\mathcal{X}) \). \( \Box \)

There is the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\Gamma_+} & \text{mod}^\sim(R) \\
\downarrow{\sim} & & \downarrow{\tilde{\cdot}} \\
\text{coh}(\mathcal{X}) & \xrightarrow{T} & \frac{\text{mod}^\sim(R)}{\text{mod}_0^\sim(R)}
\end{array}
\]

where \( T \) is the quotient functor and \( T \circ \Gamma_+ \) gives an equivalence. Identifying \( \mathcal{H} \) with \( \text{coh}(\mathcal{X}) \) via the equivalence \( \tilde{\cdot} \circ \Gamma_+ \), it is clear that the efficient automorphism \( \sigma \) (defining \( \tilde{R} \)) corresponds to the degree shift on \( \text{mod}^\sim(R) \), \( X \mapsto X(1) \). Moreover, \( \tilde{R}(n) = L(n) \).

We get the following statement which is a generalization of the statement which sometimes is referred to as Grothendieck’s lemma if \( \mathcal{X} = \mathbb{P}^1(\mathbb{K}) \).

**Corollary 2.5.4.** Each indecomposable vector bundle over \( \mathcal{X} \) is isomorphic to \( L(n) \) or to \( \mathbf{L}(n) \) for some \( n \in \mathbb{Z} \). \( \Box \)
Here, the number $n$ is unique in orbit cases I and III; in orbit case II one has $\mathcal{L} = L(1)$. Moreover, $\mathcal{L}$ is not a line bundle precisely in orbit case I.
CHAPTER 3

Tubular shifts and prime elements

This chapter is motivated by the question when two tubular shifts \( \sigma_x \) and \( \sigma_y \) coincide. Whereas the answer over an algebraically closed field is trivial, this question becomes interesting over an arbitrary field and we will show that it is linked to (the centre of) the function field.

In Chapter 1 we got the graded factoriality of \( R = \Pi(L, \sigma) \) (with \( \sigma \) efficient) by showing that there is a one-to-one correspondence between points \( x \in X \) and prime elements \( \pi_x \) (up to multiplication with some unit) naturally given by forming universal extensions. In this chapter we show that this actually extends to a natural correspondence between prime elements \( \pi_x \) and tubular shifts \( \sigma_x \) together with the natural transformations \( 1_{\mathcal{H}} \xrightarrow{x} \sigma_x \); in fact, such a transformation is induced by right multiplication with \( \pi_x \).

As a consequence we get a relation between the tubular shifts and the degree shift. The difference between tubular shifts and the degree shift is given by ghosts arising from prime elements which are not central (up to multiplication with a unit). Thus we get important information on the structure of the Picard group. In particular, this group is not always torsionfree.

3.1. Central prime elements

Let \( X \) be a homogeneous exceptional curve with structure sheaf \( L \) and \( \sigma \) be an efficient automorphism. Let \( R = \Pi(L, \sigma) \) be the corresponding orbit algebra. Recall, that we denote \( \sigma^n(F) \) also by \( F(n) \) for \( F \in \mathcal{H} \) and also for \( F \in \text{mod}^n R \) (degree shift).

3.1.1 (Central multiplication). It is convenient to consider first the special, central case. Let \( r \in R \) be a central homogeneous element of degree \( n \). Then multiplication from the right with \( r \) induces a homomorphism \( M \xrightarrow{\cdot r} M(n) \) for each \( M \in \text{mod}^n R \) (right modules), and by sheafification, this induces also a morphism \( \tilde{M} \xrightarrow{\cdot r} \tilde{M}(n) \) in \( \mathcal{H} \). We obtain a natural transformation \( 1_{\mathcal{H}} \xrightarrow{\cdot r} \sigma^n \). It follows easily that

- for \( M = R \) we get \( \cdot r = r \in \text{Hom}(L, L(n)) \);
- if we apply the functor \( \Gamma_+ \) to \( F \xrightarrow{\cdot r} F(n) \) in \( \mathcal{H} \), then we get back multiplication with \( r \) on the level of graded right \( R \)-modules.
3. TUBULAR SHIFTS AND PRIME ELEMENTS

Theorem 3.1.2. Let $R = \Pi(L, \sigma)$ with $\sigma$ being efficient. Let $x \in X$ such that the prime element $\pi_x \in R$ is central of degree $d$. Then there is a natural isomorphism from the tubular shift $\sigma_x$ to the degree shift $\sigma^d$, which is compatible with the natural transformations $1_H \xrightarrow{\pi_x} \sigma_x$ and $1_H \xrightarrow{\sigma^d} \sigma_x$.

Proof. Let $M \in \mathcal{H}_+$ be of rank $r > 0$. Let $S = S_x$ be the simple object concentrated in $x$. There is the universal exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha_M} \sigma_x(M) \longrightarrow \text{Ext}^1(S, M) \otimes_{\text{End}(S)} S \longrightarrow 0.$$ 

The morphism $M \xrightarrow{\pi_x} M(d)$ is a monomorphism and its cokernel is concentrated in the point $x$ and of length $r \cdot e(x)$: using a line bundle filtration of $M$ and induction, it suffices to show this for line bundles $L'$. If $L'$ is a shift of $L$ then the cokernel is $S^e_x(x)$. In orbit case III we have to consider also the case $L' = L$. There is an irreducible map $L \xrightarrow{u} L$, and the cokernel $S_0$ is a simple object. One can assume that $S_0 \not\cong S_x$. It then follows that the cokernel of the map $L \xrightarrow{\pi_x} L(d)$ is also isomorphic to $S^e_x(x)$.

We have to show that the cokernel $C$ of $M \xrightarrow{\pi_x} M(d)$ is semisimple: The map $C \xrightarrow{\pi_x} C(d)$ is zero, which follows from the commutative exact diagram

$$0 \longrightarrow M \xrightarrow{\pi_x} M(d) \xrightarrow{\pi_x(d)} M(2d) \xrightarrow{\pi_x} C \longrightarrow 0.$$ 

Applying the exact functor $\phi \circ q_x$ from 2.2.6 we get a short exact sequence of right $R_P$-modules:

$$0 \longrightarrow M_P \xrightarrow{\pi_x} M(d)_P \longrightarrow C_P \longrightarrow 0.$$ 

Since the map $C \xrightarrow{\pi_x} C(d)$ is zero, we see that $C_P$ is a graded $R_P/P_P$-module, hence semisimple by 2.2.8. It follows that $C$ is a direct sum of copies of $S$.

Hence we get a commutative, exact diagram

$$0 \longrightarrow M \xrightarrow{\alpha_M} M(x) \xrightarrow{\beta_M} M_x \longrightarrow 0$$

with isomorphisms $i_M$ and $j_M$ (compare 0.4.2 (2)). With the uniqueness property 0.4.2 (1) of $\sigma_x(f)$ it follows easily that for each $f \in \text{Hom}(M, N)$ (with $N \in \mathcal{H}_+$) we have $\sigma_x(f) \circ i_M = i_N \circ f(d)$. Therefore, the functors $\sigma^d$ and $\sigma_x$ are naturally isomorphic on $\mathcal{H}_+$. (Actually, the argument shows, that this holds everywhere outside $U_x$.) With Lemma 1.2.2, presenting each
object in \( \mathcal{H}_0 \) as cokernel of a monomorphism between objects from \( \mathcal{H}_+ \), the result follows by diagram chasing.

\[ \square \]

**Corollary 3.1.3.** Let \( x, y \in \mathbb{K} \) such that the corresponding prime elements \( \pi_x \) and \( \pi_y \) are central of the same degree in \( R \). Then the tubular shifts \( \sigma_x \) and \( \sigma_y \) are isomorphic.

\[ \square \]

### 3.2. Non-central prime elements and ghosts

**3.2.1 (Normal multiplication).** We generalize Theorem 3.1.2 to arbitrary prime elements. Therefore, assume that \( r \in R \) is a (non-zero) normal element of degree \( n \). Then \( r \) induces the automorphism \( \gamma = \gamma_r \) of the graded algebra \( R \), by the formula \( sr = r\gamma(s) \). Let \( M \in \text{mod}^\pi(Z) \). Denote by \( M_\gamma \) the object in \( \text{mod}^\pi(Z) \), where \( M_\gamma \) and \( M_\gamma \) coincide as abelian groups, and where the \( R \)-action on \( M_\gamma \) is defined by

\[
(m \cdot r) \cdot s \overset{def}{=} m \cdot (r \cdot \gamma(s)).
\]

Then multiplication \( r : m \mapsto m \cdot r \) defines a morphism between the graded right modules \( M \) and \( M_\gamma(n) \), since

\[
(m \cdot s) \cdot r = m \cdot (s \cdot r) = m \cdot (r \cdot \gamma(s)) = (m \cdot r) \cdot \gamma s.
\]

Since obviously \( \gamma(m) = m \), the algebra automorphism \( \gamma \) gives rise to an automorphism \( \gamma^*(M) = M_\gamma \) on \( \text{mod}^\pi(Z) \) and (denoted by the same symbol) also on \( \mathcal{H} = \frac{\text{mod}^\pi(Z)}{\text{mod}^0(Z)} \), such that \( \sigma \circ \gamma^* = \gamma^* \circ \sigma \); moreover, \( \gamma^*(\tilde{M}) = \tilde{M}_\gamma \). Sheafification gives a homomorphism \( \tilde{M} \to \tilde{M}_\gamma(n) \), yielding a natural transformation \( 1_{\mathcal{H}} \to \sigma^n \circ \gamma^* = \gamma^* \circ \sigma^n \). Obviously, \( \gamma \) gives rise to an isomorphism \( R \to \tilde{R}_\gamma \) of graded right \( R \)-modules, hence \( \gamma^* \) leaves \( L \) fixed, that is, \( \gamma^* \in \text{Aut}(\mathbb{K}) \). Note that after identifying \( R \) with \( \tilde{R}_\gamma \) via \( \gamma \), the functor \( \gamma^* \) on \( \text{mod}^\pi(Z) \) acts on elements \( f \cdot \text{Hom}(R,R(m)) \) (in contrast to \( \text{Hom}(R,\tilde{R}_\gamma(m)) \)) like \( \gamma^{-1} \), since \( (\gamma^{-1} \circ (\gamma^*(f)) \circ \gamma)(r) = \gamma^{-1}(f) \cdot r \). We will also make use of the notation \( \gamma_* = (\gamma^{-1})^* \).

\[ \square \]

The \( k \)-algebra automorphisms induced by normal elements are special cases of a more general class of algebra automorphisms:

**Definition 3.2.2.** Let \( \gamma \) be an automorphism of the graded \( k \)-algebra \( R \). \( \gamma \) is called *prime fixing*, if for all homogeneous prime ideals \( P \) (of height one) we have \( \gamma(P) = P \). In other words, \( \gamma \) is prime fixing if and only if for each prime element \( \pi \) there is a unit \( u \in R_\pi \) such that \( \gamma(\pi) = \pi u \). Denote by \( \text{Aut}_0(R) \) the subgroup of \( \text{Aut}(R) \) of all graded algebra automorphisms, which are prime fixing.

\[ \square \]

Of course, the group \( \text{Im}(R) \) of inner graded \( k \)-algebra automorphisms of \( R \) is a (normal) subgroup of \( \text{Aut}_0(R) \).
Proposition 3.2.3. (1) Let $\gamma$ be a graded algebra automorphism of $R$. Then the induced automorphisms $\gamma_*$ and $\gamma^*$ on $H$ are automorphisms of $X$. Moreover, $\gamma^*$ is trivial (that is, isomorphic to the identity) if and only if $\gamma$ is inner. Hence the assignment $\gamma \mapsto \gamma_*$ induces an injective group homomorphism from $\text{Aut}(R)/\text{Inn}(R)$ into $\text{Aut}(X)$.

(2) Let $\gamma$ be a prime fixing automorphism. Then the induced automorphisms $\gamma_*$ and $\gamma^*$ on $H$ are elements of the ghost group. Hence, there is an injective group homomorphism from $\text{Aut}_0(R)/\text{Inn}(R)$ into the ghost group $G$.

(3) Let $r \in R$ be normal. The associated algebra automorphism $\gamma = \gamma_r$ is prime fixing, and it is inner if and only if there is some unit $u \in R_0^*$ such that $ru$ is central.

Proof. (1) As already seen above, $\gamma^*$ fixes $L$. The automorphism $\gamma$ is inner if and only if $\gamma^*$ on $\text{mod}^Z(R)$ is isomorphic to the identity [7, II.5]. In this case, also $\gamma^*$ on $H$ is isomorphic to the identity. The converse follows with 2.1.7, since $\gamma^* \simeq 1_H$ induces a bimodule isomorphism $R \simeq R_{\gamma}$, which implies that $\gamma$ is inner [7, II.5].

(2) Let $x \in X$ and $\pi_x$ be the associated prime element of degree $d$. By assumption, there is some $u \in R_0^*$ such that $\gamma(\pi_x) = \pi_x u$. Then $L \xrightarrow{\pi_x} L(d)$ induces (via $\Gamma_+$) the commutative, exact diagram of right graded $R$-modules

\[
0 \longrightarrow R \xrightarrow{\pi_x} R(d) \xrightarrow{\gamma} R/\pi_x R \longrightarrow 0
\]

and sheafification implies (using $\widetilde{\cdot} \circ \Gamma_+ = 1_H$)

\[
(3.2.1) \quad 0 \longrightarrow L \xrightarrow{\pi_x} L(d) \xrightarrow{\gamma} L_x \longrightarrow 0
\]

Since $\gamma^*$ fixes $L_x = S^e_x(x)$, the point $x$ is fixed. It follows that $\gamma^*$ is a ghost.

(3) From almost commutativity 1.6.4 it follows, that $\gamma_r$ is prime fixing. The rest is clear. \qed

In particular ghosts are obtained in the following way.

Corollary 3.2.4. Let $r \in R$ be normal. Assume that for all units $u \in R_0^*$ the element $ru$ is not central. Then for $\gamma = \gamma_r$ we have $\gamma^* \not\simeq 1_H$, thus $\gamma^*$ is a ghost. \qed
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The proof of statement (2) in Proposition 3.2.3 shows that there is the following more general fact.

**Proposition 3.2.5.** Let \( \gamma \) be a graded algebra automorphism of \( R \). Let \( x, y \) be points in \( X \) with corresponding homogeneous prime ideals \( P_x, P_y \) and simple objects \( S_x, S_y \), respectively. If \( \gamma(P_x) = P_y \) then \( \gamma^*(S_x) \cong S_y \). In particular, \( \gamma \) is prime fixing if and only if \( \gamma^* \) is point fixing. \( \square \)

**Definition 3.2.6.** An element \( \gamma \) of \( \text{Aut}(X) \) is called liftable to \( R \), if there is a graded \( k \)-algebra automorphism \( \phi \) of \( R \) such that \( \phi^* \) represents \( \gamma \).

Thus, a ghost \( \gamma \) is liftable to \( R \) if and only if its class in \( G \) lies in the image of the injective homomorphism \( \text{Aut}_0(R)/\text{Inn}(R) \longrightarrow G \).

**Problem 3.2.7.** Is any ghost liftable to \( R \)? Is \( \text{Aut}_0(R) \) up to inner automorphisms generated by all the \( \gamma_r, r \in R \) (normal/prime)?

The main result of this chapter is the following.

**Theorem 3.2.8.** Let \( R = \Pi(L, \sigma) \) with \( \sigma \) being efficient. Let \( y \in X \) with associated prime element \( \pi_y \in R \) of degree \( d \). Let \( \gamma = \gamma_{\pi_y} \) be the associated algebra automorphism of \( R \) and \( \gamma^* \) the induced automorphism of \( \mathcal{H} \). Then there is a natural isomorphism from the tubular shift \( \sigma_y \) to \( \sigma^d \circ \gamma^* \), which is compatible with the natural transformations \( 1_{\mathcal{H}y} \longrightarrow \sigma_y \) and \( 1_{\mathcal{H}} \cdot \pi_y \longrightarrow \sigma_y \circ \gamma^* \).

**Proof.** Consider the proof of the preceding lemma for \( r = \pi_y = \pi_x \) (hence \( u = 1 \)). Using the identity \( \pi_y \cdot s = \gamma^{-1}(s) \cdot \pi_y \) for all \( s \in R \), the lower exact sequence in the diagram (3.2.1) induces the commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \gamma^*(L) & \xrightarrow{\pi_y^*} & \gamma^*(L)(d) & \longrightarrow & \gamma^*(L_y) & \longrightarrow & 0 \\
0 & \xrightarrow{\cong} & L & \xrightarrow{-\pi_y} & \gamma^*(L)(d) & \xrightarrow{\cong} & L_y & \longrightarrow & 0,
\end{array}
\]

and hence \( L \xrightarrow{-\pi_y} \gamma^*(L)(d) \) yields an \( S_y \)-universal extension. The rest of the proof is completely analogue to the proof of 3.1.2. \( \square \)

**Corollary 3.2.9.** Let \( G' \) be the subgroup of the ghost group \( G \) generated by the automorphisms \( \gamma_{\pi_x}^* (x \in X) \). Then \( \text{Pic}(X) \subset \langle \sigma \rangle \times G' \). \( \square \)

**Corollary 3.2.10.** If \( m \) is a positive integer such that \( \pi_y^m \) is central up to multiplication with some unit, then \( \sigma_y^m \) is naturally isomorphic to \( \sigma^m \).

**Proof.** By the assumption, \( \gamma^m \) is an inner automorphism on \( R \). \( \square \)

**Corollary 3.2.11.** Let \( x_1, \ldots, x_s \in X \) and \( \pi_{x_1}, \ldots, \pi_{x_s} \in R \) corresponding primes. Denote by \( d_i = e(x_i)f(x_i)/\ell \) the degree of \( \pi_{x_i} \). Let \( m_1, \ldots, m_s \) be integers. The following are equivalent:
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(1) \( \sigma_x^m \circ \cdots \circ \sigma_x^{m_s} \simeq 1_H \).

(2) \( \sum_{i=1}^s m_i d_i = 0 \) and the graded automorphism \( \gamma_{x_1}^{m_1} \circ \cdots \circ \gamma_{x_s}^{m_s} \) of \( R \) is inner.

(3) There is a unit \( u \in R \) such that the element \( u \gamma_{x_1}^{m_1} \cdots \gamma_{x_s}^{m_s} \) is a central element of degree zero in the graded quotient division ring \( \text{Quot}^Z(R) \).

If, moreover, there is a central unit of degree one in \( \text{Quot}^Z(R) \), then these conditions are also equivalent to

(4) There is a unit \( u \in R \) such that the element \( u \gamma_{x_1}^{m_1} \cdots \gamma_{x_s}^{m_s} \) lies in the centre of the function field \( k(X) \).

Note that we get as a special case a criterion for two tubular shifts \( \sigma_x \) and \( \sigma_y \) to be isomorphic.

**Proof.** By Theorem 3.2.8 we have

\[
\sigma_x^{m_1} \circ \cdots \circ \sigma_x^{m_s} \simeq \sigma \prod_{i=1}^s m_i d_i \circ (\gamma_{x_1}^{*})^{m_1} \circ \cdots \circ (\gamma_{x_s}^{*})^{m_s}.
\]

(Note, that the \( \gamma_{x_i}^{*} \) commute with all \( \gamma_{x_j}^{*} \) and all \( \sigma_{x_j} \) by 0.4.8 and 1.6.2.)

It follows from Proposition 3.2.3 that if \( r \) and \( s \) are normal, then \( \gamma_{x}^{*} \simeq \gamma_{s}^{*} \) if and only if there is a unit \( u \in R \) such that \( \gamma_{ru} = \gamma_{s} \). This shows the equivalence of (1) and (2).

The equivalence of (2) and (3) is clear.

For the equivalence of (3) and (4) note that by the existence of the central unit of degree one we have \( \text{Quot}^Z(R) = k(X)[T, T^{-1}] \), where \( T \) is a central variable of degree one. Then an element which lies in the centre of \( k(X) \) also lies in the centre of \( \text{Quot}^Z(R) \). \( \square \)

**Remark 3.2.12.** It remains the question whether there is always a central unit of degree one in \( \text{Quot}^Z(R) \). This leads to Problem 1.1.12.

(a) Let \( R = \Pi(L, \sigma) \) with \( \sigma \) being efficient. It follows from (3.2.2) that there exists an efficient tubular shift if and only if there is a prime (or normal) element of degree one. It follows that in the example in 1.1.13 (4) there is no prime element in \( R \) of degree one.

(b) It follows from Theorem 3.2.8 that there exists an efficient automorphism lying in the Picard group if and only if the positive integers \( e(x)f(x)/\ell \) \( (x \in X) \) generate the group \( \mathbb{Z} \).

(c) Assume that the efficient automorphism \( \sigma \) lies in the Picard group, say \( \sigma \overset{\text{def}}{=} \prod_{i=1}^s \sigma_{x_i}^{m_i} \) (with \( m_i \) non-zero integers), and let \( R = \Pi(L, \sigma) \). It follows from (3.2.2) that \( \prod_{i=1}^s (\gamma_{x_i}^{*})^{m_i} \simeq 1_H \). Hence \( \prod_{i=1}^s \gamma_{x_i}^{m_i} \) is an inner automorphism of \( R \). From this it follows easily that there is a unit \( u \in R \) such that the element \( u \prod_{i=1}^s \pi_{x_i}^{m_i} \) is a central unit of degree one in \( \text{Quot}^Z(R) \). Thus, if Problem 1.1.12 would have a positive answer then we would not need the extra condition in Corollary 3.2.11.
We conclude the chapter by reformulating some results in the language of divisors.

3.2.13 (Divisors). (1) Denote by $N_0^*$ the group of non-zero elements in $k(X)$ which are fractions of normal elements in $R$ (of the same degree). Denote by $\text{Div}(X)$ the abelian group of all formal sums of the form $\sum_{x \in X} m_x x$ where $m_x \in \mathbb{Z}$, almost all zero. It follows from 1.6.3 that there is an exact sequence of groups

$$1 \longrightarrow R_0^* \rightarrow N_0^* \xrightarrow{\text{div}} \text{Div}(X) \xrightarrow{\deg_e} \mathbb{Z},$$

where $\deg_e(x) = e(x)f(x)/\ell$ and where div is induced by the correspondence $R\pi_x \mapsto x$ from 1.5.2.

(2) Assume that $\deg_e : \text{Div}(X) \rightarrow \mathbb{Z}$ is surjective. Denote by $Z_0^*$ the subgroup of $N_0^*/R_0^*$ given by the classes which admit representatives lying in the centre of $k(X)$. The preceding discussion has shown that there is an exact sequence of abelian groups

$$0 \longrightarrow Z_0^* \longrightarrow N_0^*/R_0^* \longrightarrow \text{Pic}_0(X) \longrightarrow 0.$$

Examples 3.2.14. (1) The twisted polynomial case. Let $R = F[X; Y, \alpha]$, let $x$ and $y$ be the points corresponding to the primes $X$ and $Y$, respectively. Since any prime ideal of height one different from $RY$ is generated by a central element, for any points $z_1, z_2$ different from $y$ we have $\sigma_{z_1} \simeq \sigma_{z_2}$ if and only if $d(z_1) = d(z_2)$. If $\alpha$ is not inner, then $\sigma_x$ and $\sigma_y$ are not isomorphic.

(2) The square roots case. Let $R = \mathbb{Q}\langle x, y, z \rangle$ be the graded factorial algebra from 1.7.12. It is easy to see that $x$ (central), $y$ and $z$ are (up to multiplication with units) the only prime elements of degree one (using linear independence of $x$, $y$, $z$ and Lemma 1.6.2). So there are precisely three unirational points, and the associated tubular shifts are pairwise non-isomorphic.

Further applications of the results of this section are discussed in Section 5.4 and in 5.7.2.
CHAPTER 4

Commutativity and multiplicity freeness

In this chapter we characterize the exceptional curves \( X \) which are commutative. We show that \( X \) is commutative if and only if it is multiplicity free. The proof of this is an application of the graded factoriality.

4.1. Finiteness over the centre

Before we characterize those exceptional curves which are commutative, we remark that exceptional curves in general are close to commutativity in the sense that they have a “large” centre. On the other hand, in the following sections it will be pointed out that they are commutative only in very special cases.

Let \( X \) be a homogeneous exceptional curve. It is well-known that the function field \( k(X) \) is of finite dimension over its centre \([6]\). A similar result is true for the orbit algebras.

Let \( \sigma \) be efficient and \( R = \Pi(L, \sigma) \).

**Proposition 4.1.1.** Let \( T = L \oplus L \) and \( S = \Pi(T, \sigma) \). Then \( S \) is module-finite over its noetherian centre.

**Proof.** By \([10]\) it is sufficient to show that \( S \) is a semiprime noetherian PI ring of global dimension two. This follows as in \([6, \text{Thm. 6.5}]\). \( \square \)

**Corollary 4.1.2.** Let \( R = \Pi(L, \sigma) \). Then \( R \) is module-finite over its noetherian centre.

**Proof.** Let \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), which is an element of degree zero in \( S = \Pi(T, \sigma) \) with \( e^2 = e \). Moreover, \( R \simeq eSe \subset S \). Let \( C \) be the centre of \( S \). Then \( eCe \) is commutative, noetherian and lies in the centre of \( eSe = R \). Since \( S \) is a finitely generated \( C \)-module by the proposition, \( R \) is finitely generated over \( eCe \). Then \( R \) is also finitely generated over its centre. \( \square \)

**Corollary 4.1.3.** Let \( C \) be the centre of \( R \). The assignment \( P \mapsto P \cap C \) is a bijection from the homogeneous prime ideals of height one in \( R \) onto the homogeneous prime ideals of height one in \( C \).

**Proof.** We refer to graded versions of results in the literature: The surjectivity of the map follows by general properties of finite centralizing
extensions [76, 10.2]. The injectivity follows from [35, 11.20] since in our situation cliques of prime ideals of height one are singletons.

Problem 4.1.4. What is the geometric interpretation of this map? What is the structure and explicit form of the centre?

See [17] for another approach studying this centre.

Corollary 4.1.5. The centre of $R$ is a graded normal domain of Krull dimension two.

Proof. It is sufficient to show that $C$ is graded normal. Let $rs^{-1}$ be a homogeneous element in the graded quotient field of $C$, which is a subfield of the graded quotient ring $RX^{-1}$, where $X$ is the multiplicative set of normal elements. Then $r$ and $s$ are normal elements, and we cancel common prime factors in $r$ and $s$ and get a fraction $rs^{-1} = r's^{-1} \in RX^{-1}$, where $r'$ and $s'$ have no common prime factor. If $rs^{-1}$ is integral over $C$, then we see that $s'$ has no prime factor, hence is a unit in $R$, and $rs^{-1} = r' \in R$. We get $r = r's$, and since $r$ and $s$ are central, also $r'$ is central, that is, $r' \in C$: For each $x \in R$ we have

\[
x r's = xr = rx = r'sx = r'xs,
\]

and $s$ can be canceled.

4.2. Commutativity of the coordinate algebra

If $k$ is not algebraically closed it happens very rarely that the orbit algebra $\Pi(L, \sigma)$ is commutative\(^1\). It is shown in [53] (if the characteristic of $k$ is different from two) that a small preprojective algebra is commutative if and only if there is some (commutative) finite field extension $K/k$ such that the tame bimodule $M$ is the Kronecker bimodule $K(K \oplus K)_K$ or a $(4,1)$- or $(1,4)$-bimodule of a skew field of quaternions over $K$. So it is commutative only in very special cases. This result carries over to our type of orbit algebras. Another proof will be given by Theorem 4.3.5.

Theorem 4.2.1. Assume that the characteristic of $k$ is different from two. Let $R = \Pi(L, \sigma)$, where $\sigma$ is efficient. Then $R$ is commutative if and only if there is a (commutative) finite field extension $K/k$ such that as graded algebras either

(a) $R \simeq K[X,Y]$, the polynomial algebra graded by total degree; or
(b) $R \simeq K[X,Y,Z]/Q$, where $Q = Q(X,Y,Z)$ is an anisotropic quadratic form over $K$.

\(^1\)Commutativity is always meant in the usual and not in the graded sense.
Moreover, in case (a) $M$ is the Kronecker bimodule $K \oplus K$ over $K$; in case (b) $M$ is the bimodule $KF_F$, where $F$ is a skew field of quaternions over $K$. 

Let $\Lambda$ be the corresponding tame bimodule algebra. If we assume that $k$ is the centre of $\Lambda$, then for the field extension $K/k$ in theorem we have $K = k$.

Recall, that $X$ is multiplicity free, if $e(x) = 1$ for all $x \in X$. We call $R$ almost commutative\(^2\), if for all homogeneous $r$, $s \in R$ there is an $\alpha \in R^*_0$ such that $rs = \alpha sr$. Since in orbit case III the $(2,2)$-bimodule is simple, each rational point $x$ (that is $f(x) = 1$), which always exists, has multiplicity $e(x) > 1$ (by 0.6.1). Hence, $X$ is never multiplicity free in orbit case III.

**Proposition 4.2.2** (Almost commutativity). $R$ is almost commutative if and only if $X$ is multiplicity free.

**Proof.** If $e(x) = 1$ for all $x \in X$ then the prime elements are just the irreducible elements by 1.6.5 and 1.6.7. Then the almost commutativity follows by Proposition 1.6.3 since each non-zero homogeneous element is a product of irreducible elements.

Conversely, if almost commutativity holds then each homogeneous prime ideal of height one (which is principal) is easily seen to be completely prime. Now apply Theorem 1.2.3 (3).

The preceding proposition will be strengthened in the following section.

### 4.3. Commutativity of the function field

As was pointed out by Ringel [88] there is a strange commutativity behaviour of the function field (see also [22]). For example ([88]) the bimodule $\mathbb{R}[H]$ with noncommutative data leads to the commutative function field

$$\text{Quot}(\mathbb{R}[U, V]/(U^2 + V^2 + 1))$$

whereas the bimodule $Q(\sqrt{2}, \sqrt{3})_{Q(\sqrt{2}, \sqrt{3})}$ with commutative data leads to the noncommutative function field given by the quotient division ring of

$$Q(U, V)/(UV + VU, V^2 + 2U^2 - 3).$$

(Compare also Proposition 1.7.12.) We will explain this effect in this section.

Recall that an exceptional curve $X$ is called commutative if its function field $k(X)$ is commutative. The main result of this section is the following theorem. (Note that we allow also the weighted case in the theorem since it makes no difference.)

\(^2\)In the literature the term is also used with another meaning [76].
Theorem 4.3.1. Let $X$ be an exceptional curve. The following statements are equivalent:

1. $X$ is commutative.
2. $X$ is multiplicity free.
3. For each rational point $x \in X$ we have $e(x) = 1$. 

In this case and if additionally $\text{char } k \neq 2$ there is some finite field extension $K/k$ such that

- $k(X) \simeq K(T)$ if the numerical type of $X$ is $\varepsilon = 1$; or
- $k(X)$ is isomorphic to the quotient field of $K[U, V]/(-aU^2 - bV^2 + ab)$ for some anisotropic quadratic form $-aX^2 - bY^2 + abZ^2$ over $K$ if the numerical type of $X$ is $\varepsilon = 2$.

Remark 4.3.2. (1) It follows (together with Theorem 5.3.4 and Proposition 5.5.1) that if $\text{char } k \neq 2$ and $X$ is commutative then the ghost group $\mathcal{G}$ is trivial.

(2) In the theorem, $K$ is the field $\text{End}(L)$ and is the centre of $\mathcal{H}$.

Since the function field and the multiplicities are preserved by insertion of weights, it is sufficient to treat the homogeneous case.

Note that in case $k$ is a finite field the equivalence $(1) \Leftrightarrow (2)$ is given by Corollary 2.3.9. The implication $(1) \Rightarrow (2)$ is given by 2.2.14. We give now another argument for this.

Proposition 4.3.3. Let $X$ be homogeneous. Let $x \in X$ be some point.

Then as graded algebras, $\Pi(L, \sigma_x) \subset k(X)[T]$, where $T$ is a central variable. In particular, if $k(X)$ is commutative, so is $\Pi(L, \sigma_x)$.

Proof. Let $\overline{\mathcal{H}} = \mathcal{H}/\mathcal{H}_0$. The natural transformation $1_{\mathcal{H}} \xrightarrow{x} \sigma_x$ induces a natural isomorphism $1_{\overline{\mathcal{H}}} \xrightarrow{\pi} \sigma_x$. Denote by $\overline{L}$ the class of $L$ in $\overline{\mathcal{H}}$. Then there is an isomorphism of graded rings

$$\text{End}_{\overline{\mathcal{H}}}(\overline{L})[T] \longrightarrow \bigoplus_{n \geq 0} \text{Hom}_{\overline{\mathcal{H}}}(\overline{L}, \sigma_x^n(\overline{L})), \ fT^n \mapsto f * \overline{\sigma}_x^n$$

(where on the right hand side the multiplication and the power of $\overline{\sigma}_x^n$ are taken in the orbit algebra sense), and $\Pi(L, \sigma_x)$ embeds naturally. \hfill $\square$

Corollary 4.3.4. Let $X$ be homogeneous. Assume that the function field is commutative. Then $X$ is multiplicity free. Moreover, there is some unirational point $x$ such that the orbit algebra $\Pi(L, \sigma_x)$ is commutative graded factorial.

Proof. Take any point $x$ and form the orbit algebra $R = \Pi(L, \sigma_x)$ with respect to the (not necessarily efficient) associated tubular shift $\sigma_x$. By the preceding proposition $R$ is commutative. Moreover, by Serre’s theorem 2.1.2, $R$ is a homogeneous coordinate algebra for $X$, and hence “classical” algebraic
geometry shows that \( X \) is multiplicity free (see for example [54]). In particular, there exists some unirational point \( x \) and the assertion follows since \( \sigma_x \) is exhaustive.

The orbit algebras \( R = \Pi(L, \sigma_x) \), where \( \sigma_x \) is exhaustive and where \( R \) is commutative, are described (in case \( \text{char} k \neq 2 \)) in Theorem 4.2.1. From this we get the explicit form of the function fields as in Theorem 4.3.1. This explicit description follows again from the next theorem which provides also the proof for the implication \((3) \Rightarrow (1)\) in Theorem 4.3.1.

**Theorem 4.3.5.** Let \( X \) be homogeneous. Assume that for all rational points \( x \in X \) we have \( e(x) = 1 \). Then for each rational point \( x \) the orbit algebra \( \Pi(L, \sigma_x) \) is commutative.

Moreover, if \( \varepsilon = 1 \), then there is a finite field extension \( K/k \) such that \( \Pi(L, \sigma_x) \simeq K[X,Y] \), where \( X \) and \( Y \) are central variables of degree one. If \( \varepsilon = 2 \) and \( \text{char}(k) \neq 2 \), then there is a finite field extension \( K/k \) such that \( \Pi(L, \sigma_x) \simeq K[X,Y,Z]/(-aX^2 - bY^2 + abZ^2) \), with \( a, b \in K^* \) such that \(-aX^2 - bY^2 + abZ^2\) is an anisotropic quadratic form over \( K \).

**Proof.** There is a rational point \( x \). By assumption \( x \) is unirational. Thus the associated tubular shift \( \sigma_x \) is efficient and the orbit algebra \( R = \Pi(L, \sigma_x) \) graded factorial. We have \( R = R_0 \langle R_1 \rangle \) and \([R_1 : R_0] = \varepsilon + 1\). Moreover, \([R_n : R_0] = \varepsilon n + 1\). (Note that by Corollary 0.6.2 in case \( \varepsilon = 1 \) the underlying tame bimodule \( M \) is non-simple.)

**Fact:** Let \( u \) be a non-zero element of \( R \), homogeneous of degree one. Then \( u \) is prime.

Namely, \( u \) is irreducible, hence by 1.3.3 a divisor of some prime element \( \pi \), which because of \( \text{deg}(u) = 1 \) is associated to a rational, hence multiplicity free point. It follows from 1.6.5 that \( u \) equals \( \pi \) up to some unit and hence is prime itself.

It follows that there are rational points \( y \) (and \( z \)) such that the prime elements \( \pi_x, \pi_y \) (and \( \pi_z \)) are linearly independent over \( R_0 \) and \( R = R_0 \langle \pi_x, \pi_y \rangle \) (in case \( \varepsilon = 1 \)) or \( R = R_0 \langle \pi_x, \pi_y, \pi_z \rangle \) (in case \( \varepsilon = 2 \)). Moreover, by 1.7.1 the prime \( \pi_x \) is central. We have to show

(a) \( R_0 \) is commutative;
(b) \( \pi_y \) (and \( \pi_z \)) commutes with each element from \( R_0 \);
(c) \( \pi_y \pi_z = \pi_z \pi_y \) (in case \( \varepsilon = 2 \)).

We only discuss the case \( \varepsilon = 2 \) since the arguments for the case \( \varepsilon = 1 \) are similar and even easier. Since \( \pi_y \) and \( \pi_z \) are prime there is a unit \( \alpha \in R_0^* \) such that \( \pi_y \pi_z = \alpha \pi_z \pi_y \) (by 1.6.2). By the Fact above, \( \pi_x + \pi_y \) is prime, therefore
commutes with the prime \( \pi_z \) up to a unit, and \( \alpha = 1 \) follows. Let \( f \in R_0, \ f \neq 0 \). For \( a \in R_0 \), again by using the Fact above and considering the product 
\((\pi_x + a\pi_y)f\) it follows that \( fa = a\gamma_y(f) \), where \( \gamma_y \) is the automorphism 
induced by the normal element \( \pi_y \). In particular, for \( a = 1 \) we get \( \gamma_y(f) = f \). 
It follows that \( R_0 \) is commutative and that \( \pi_y \) is central. Similarly, \( \pi_z \) is central. 
Thus, \( R \) is commutative.

With \( K = R_0 \) we get 
\( R \cong K[X, Y] \) in case \( \varepsilon = 1 \) and \( R \cong K[X, Y, Z]/Q \), 
where \( Q \) is a homogeneous quadratic polynomial, in case \( \varepsilon = 2 \). Assume 
\( \text{char}(k) \neq 2 \). Then by factoriality \( Q \) is anisotropic over \( K \) and hence can be 
assumed to be of the form stated in the theorem. It follows also that the 
bimodule \( M \) is given as stated in Theorem 4.2.1. \( \square \)

From this, Theorem 4.3.1 follows immediately since \( k(X) \) is the quotient 
division ring (of fractions of homogeneous elements of the same degree) of 
\( \Pi(L, \sigma_x) \).

**Lemma 4.3.6.** Let \( X \) be homogeneous and \( x \in X \) a rational point. Then 
\[ e(x) \leq \varepsilon \cdot [\text{End}(L) : k]. \]

**Proof.** Since \( x \) is rational we have \( [\text{Ext}^1(S, L) : k] = \varepsilon \cdot [\text{End}(L) : k] \), 
hence 
\[ e(x) = \frac{[\text{Ext}^1(S, L) : k]}{[\text{End}(S) : k]} = \frac{\varepsilon \cdot [\text{End}(L) : k]}{[\text{End}(S) : k]} \leq \varepsilon \cdot [\text{End}(L) : k]. \]

Note that the curve may be defined over a field which is larger than \( k \) 
(for example, over the centre of the corresponding bimodule algebra) and 
that the formula also holds with this larger field instead of \( k \).

**Example 4.3.7.** Let \( M \) be the \( \mathbb{Q} \)-\( \mathbb{Q}^{\sqrt{2}, \sqrt{3}} \)-bimodule \( \mathbb{Q}^{\sqrt{2}, \sqrt{3}} \). Let 
\( x \) be a unirational point and \( R = \Pi(L, \sigma_x) \). By 1.7.12 
\[ R = \mathbb{Q}(X, Y, Z)/(XY - YX, XZ - ZX, YZ + ZY, Z^2 + 2Y^2 - 3X^2). \]
Since this algebra is not commutative it follows from Theorem 4.3.5 that 
there is a rational point of multiplicity greater than one. We determine 
such a point explicitly. Denote the images of \( X, Y \) and \( Z \) by \( x, y \) and \( z \), 
respectively. Then \( x \) is central, \( y \) and \( z \) are normal (but not central). The 
element \( u = x - y \) is irreducible but not normal, hence not prime. But \( u \) is 
divisor of a prime \( \pi = \pi_p \). Then \( p \) is a rational point with \( e(p) > 1 \).

More precisely, (up to multiplication with a unit in \( \mathbb{Q}^* \)) we have \( \pi = x^2 - y^2 \): In fact, by the preceding lemma we have \( e(p) = 2 \), hence \( \deg(\pi) = 2 \). 
By Proposition 1.3.3 there is \( v \in R \) (of degree 1) such that \( vu = \pi \). Moreover, 
\( x^2 - y^2 = (x - y) \cdot (x + y) \) lies in the centre of \( R \). (Note that it follows easily 
from \( z(x - y) = (x + y)z \) that \( x - y \) and \( x + y \) induce isomorphic simple
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Then \( v(x^2 - y^2) = vu(x + y) = \pi(x + y) \). Since \( \pi \) is prime, it follows that \( \pi \) divides \( x^2 - y^2 \), and since both have the same degree they are associated.

**Corollary 4.3.8.** Let \( X \) be homogeneous and \( R = \Pi(L, \sigma) \) with \( \sigma \) efficient. The following are equivalent:

1. \( k(X) \) is commutative.
2. \( R \) is commutative.
3. For each homogeneous prime ideal \( P \) in \( R \) of height one the localization \( R_P^0 \) is a local (equivalently, a semiperfect) ring.

**Proof.** Recall Proposition 4.2.2 and Proposition 2.2.15.

**Problem 4.3.9.** Let \( X \) be a homogeneous exceptional curve. Is the category \( \mathcal{H} = \text{coh}(X) \) uniquely determined (up to equivalence) by the function field \( k(X) \)? This is only clear for commutative function fields.

In the context of this section Problem 2.3.10 becomes interesting again:

**Problem 4.3.10.** Find a formula for the skewness \( s(X) \) in terms of the multiplicity function \( e \). From such a formula Theorem 4.3.1 should be derived as a special case.

**Remark 4.3.11.** In case \( s(X) = 2 \) the existence of a rational point \( x \) such that \( e(x) = s(X) \) follows directly from Theorem 4.3.1.

**Remark 4.3.12.** The function field and the multiplicities are also related by a fundamental exact sequence. For this sequence one has to consider the Grothendieck category \( \text{Qcoh}(X) = \frac{\text{Mod}^{\text{fg}}(R)}{\text{Mod}^{\text{fg}}(\Lambda)} \), the quotient category modulo the Serre subcategory formed by the \( \mathbb{Z} \)-graded torsion modules.

The injective hull \( Q \) of the line bundle \( L \) is a generic sheaf (corresponding to the generic \( \Lambda \)-module, where \( \Lambda \) is the associated bimodule algebra). Moreover, \( Q \) is the injective hull of each line bundle and the endomorphism ring \( \text{End}(Q) \) is the function field \( k(X) \). (Compare [55, Lemma 14].)

For each \( x \in X \) denote by \( S_x^\omega \) the Prüfer sheaf, which is the direct limit of all \( S_x^{(n)} \) (the indecomposable sheaf of length \( n \) concentrated in \( x \)).

There is the short exact sequence in \( \text{Qcoh}(X) \)

\[
0 \longrightarrow L \longrightarrow Q \longrightarrow \bigoplus_{x \in X} \bigoplus_{e(x)} S_x^\omega \longrightarrow 0
\]

involving the multiplicities \( e(x) \). This sequence already appeared in [88, Prop. 5.2].
CHAPTER 5

Automorphism groups

Knowledge of the automorphism group of a curve contributes enormously
to a better understanding of the geometry. This chapter is devoted to the
determination of automorphism groups. By the mechanism of insertion of
weights this problem will be reduced essentially to the homogeneous case
(see Section 6.3), which we treat now.

The group $\text{Aut}(\mathcal{H})$ consists essentially of $\text{Pic}(\mathcal{X})$ and $\text{Aut}(\mathcal{X})$, where the
latter group is given by the geometric automorphisms and the ghosts. Over
an algebraically closed field the Picard group is just $\mathbb{Z}$, but this does not
hold in general, as will be shown in Theorem 5.4.1. So far, the best theorem
on the structure of $\text{Pic}(\mathcal{X})$ we have is Theorem 3.2.8, which also gives a hint
as to what ghosts look like.

In Proposition 5.1.4 we prove that $\text{Aut}(\mathcal{X})$ is isomorphic to the factor
group of the automorphism group of the underlying bimodule $M$ modulo
the inner bimodule automorphisms, which is useful for explicit calculations.
Sections 5.3 and 5.4 constitute the main part of this chapter, where we
analyse the case of a non-simple bimodule $M$. In this case it is easy to describe
the graded factorial coordinate algebra explicitly, which will be helpful in the
determination of $\text{Aut}(\mathcal{X})$. This graded algebra is of the form

$$F[X; Y, \alpha, \delta],$$

where $F$ is a finite-dimensional skew field, $X$ is a central variable, $\alpha \in
\text{Aut}(F/k)$ and $\delta : F \rightarrow F$ is a $(\alpha, 1)$-derivation such that for all $f \in F$ the
following relation holds

$$Yf = \delta(f)X + \alpha(f)Y.$$

We concentrate on the case $\delta = 0$, thus $R = F[X; Y, \alpha]$ is a graded twisted
polynomial algebra. If $r$ is the order of $\alpha$ modulo inner automorphisms, then
$\alpha$ induces a ghost automorphism $\alpha_\ast$ of order $r$. In the special case where $\alpha$
induces a generator of the Galois group $\text{Gal}(Z/k)$ (with $Z$ being the centre
of $F$) we determine $\text{Aut}(\mathcal{X})$ explicitly. Our knowledge of the prime spectrum
of $R$ can be used to distinguish ghosts from geometric automorphisms.

As an application we get a formula for the Auslander-Reiten translation.
One might expect that $\tau$ is given by degree shift by $-2$, that is, by $\sigma_x^{-2}$.
While this is true on objects, it can fail on morphisms; the correct formula
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\[ \tau = \sigma^{-1}_x \circ \sigma^{-1}_y = \sigma^{-2}_x \circ \alpha^{-1}_x. \]

We will treat also the quaternion case. Over the real numbers this is the case when \( M = \mathbb{R} \mathbb{H}, \) where \( \mathbb{H} \) is the skew field of quaternions over \( \mathbb{R} \). Together with our results on the non-simple bimodules described above we determine the geometry completely when \( k = \mathbb{R}, \) the field of real numbers. It turns out that a homogeneous exceptional curve over \( \mathbb{R} \) can be identified with a quotient of the Riemann sphere modulo an involution, possibly equipped with additional structure, and that the geometric automorphism group is just the group of directly conformal homeomorphisms on the Riemann sphere compatible with the involution and respecting the additional structure.

We conclude the chapter by exhibiting an interesting homogeneous exceptional curve over \( k = \mathbb{Q} \) whose automorphism group is isomorphic to the Klein four group.

5.1. The automorphism group of a homogeneous curve

Now we treat the automorphism group in the homogeneous case. We will see, that calculations of the automorphism group will lead us into (noncommutative) Galois theory.

5.1.1 (Bimodule automorphisms). Let \( M = F \cdot M \cdot G \) be a bimodule over the skew fields \( F \) and \( G, \) \( k \) acting centrally, with all data finite dimensional over \( k. \) We always assume \( M \neq 0. \) Define the group \( \text{Aut}(M) = \text{Aut}_k(F \cdot M \cdot G) \) to be the set of all triples \( (\varphi_F, \varphi_M, \varphi_G), \) where \( \varphi_F \in \text{Aut}(F/k), \varphi_G \in \text{Aut}(G/k), \)

\[ \varphi_M : M \rightarrow M \] is \( k \)-linear and bijective, and for all \( f \in F, \ g \in G \) and \( m \in M \) we have

\[ \varphi_M(fm, g) = \varphi_F(f) \cdot \varphi_M(m) \cdot \varphi_G(g). \]

Composition and inverse are built componentwise, the neutral element is given by \((1_F, 1_M, 1_G). \) Note that projection onto the middle component, \((\varphi_F, \varphi_M, \varphi_G) \mapsto \varphi_M \) is injective. There is an alternative description: Consider the \( k \)-category consisting of two objects with endomorphism ring \( F \) and \( G, \) respectively, and with non-zero Hom-space only in one direction, which is given by \( M. \) Then an automorphism of the bimodule \( M \) is just an autoequivalence of this category.

5.1.2 (Inner automorphisms). An element \((\varphi_F, \varphi_M, \varphi_G) \in \text{Aut}(M) \) is called \textit{inner}, if there are \( f \in F^*, \ g \in G^* \) such that for all \( x \in F, \ y \in G, \)

\( m \in M \) we have \( \varphi_F(x) = f^{-1}xf, \varphi_G(y) = y^{-1}yg \) and \( \varphi_M(m) = f^{-1}mg. \) The subgroup of all inner automorphisms is denoted by \( \text{Inn}(M) = \text{Inn}_k(F \cdot M \cdot G), \)

the factor group by \( \text{Out}(M) = \text{Out}_k(F \cdot M \cdot G) = \text{Aut}(M)/\text{Inn}(M). \)
5.2. THE STRUCTURE OF Aut(\(\mathcal{H}\))

5.2.1 (Orbit cases IIIa and IIIb). Let \(\mathcal{X}\) be a homogeneous exceptional curve with hereditary category \(\mathcal{H}\). We would like to analyse the structure of Aut(\(\mathcal{H}\)). In order to do this we have to refine the definition of orbit case III. By definition of this case there is no \(\sigma \in \text{Aut}_0(\mathcal{H})\) with \(\sigma(L) = \mathcal{T}\) (see 1.1.5). But is it possible that there is a \(\sigma \in \text{Aut}(\mathcal{H})\) with this property?
If there is such an automorphism, we call this case orbit case IIIb, if not, we call it case IIIa. In other words, in case IIIa each efficient automorphism is transitive. Of course, if in case III the endomorphism skew fields of line bundles in two different Auslander-Reiten orbits are non-isomorphic, then we are in case IIIa.

**Problem 5.2.2.** Is orbit case IIIb non-empty, that is, does there exist a tame bimodule over some field belonging to this case?

Recall that \( O \) is the \( \text{Aut}(\mathcal{H}) \)-orbit and \( O_0 \) is the \( \text{Aut}_0(\mathcal{H}) \)-orbit of \( L \). Note that only in orbit case IIIb there is a difference between \( O \) and \( O_0 \). Obviously, in any orbit case there exists a transitive automorphism: in case IIIb by definition, in all other cases there is even a point fixing one.

Fix a transitive \( \sigma \in \text{Aut}(\mathcal{H}) \) (which is unique up to an automorphism of \( X \)). Let \( \phi \in \text{Aut}(\mathcal{H}) \). Then there is a (unique) \( n \in \mathbb{Z} \) such that \( \sigma^n(\phi(L)) \simeq L \), that is, \( \sigma^n \circ \phi \in \text{Aut}(\mathcal{X}) \). In other words, each \( \phi \in \text{Aut}(\mathcal{H}) \) is a composition of a power of \( \sigma \) and some element of \( \text{Aut}(\mathcal{X}) \). Therefore, \( \text{Aut}(\mathcal{H}) \) consists of the following types of automorphisms:

- the geometric automorphisms of the curve
- the tubular shifts
- the ghosts
- one transitive automorphism (if not already in the Picard group).

As usual, denote by \( \mathcal{G} \) the ghost group of \( X \). Recall that \( \text{Pic}_0(\mathcal{X}) = \text{Pic}(\mathcal{X}) \cap \text{Aut}(\mathcal{X}) \), which is \( \text{Pic}(\mathcal{X}) \cap \mathcal{G} \).

**Proposition 5.2.3.** Let \( X \) be a homogeneous exceptional curve. Assume, that there is an exhaustive automorphism \( \sigma \) lying in \( \text{Pic}(\mathcal{X}) \), and assume that we are not in orbit case IIIb. Then there are split exact sequences of groups

\[
1 \longrightarrow \text{Pic}(\mathcal{X})/\text{Pic}_0(\mathcal{X}) \longrightarrow \text{Aut}(\mathcal{H})/\mathcal{G} \longrightarrow \text{Aut}(\mathcal{X})/\mathcal{G} \longrightarrow 1,
\]

where \( \text{Pic}(\mathcal{X})/\text{Pic}_0(\mathcal{X}) \cong \mathbb{Z} \) with generator induced by \( \sigma \), and

\[
1 \longrightarrow \langle \sigma \rangle \longrightarrow \text{Aut}_0(\mathcal{H}) \longrightarrow \mathcal{G} \longrightarrow 1.
\]

Moreover, \( \text{Aut}(\mathcal{H})/\text{Aut}_0(\mathcal{H}) \cong \text{Aut}(\mathcal{X})/\mathcal{G} \).

**Proof.** Since we exclude orbit case IIIb, \( \sigma \) is transitive. Denote the class of an automorphism \( \phi \) modulo \( \mathcal{G} \) by \( [\phi] \). It is easy to see that \( [\phi] \mapsto [\sigma^n \circ \phi] \), where \( \sigma^n(\phi(L)) \simeq L \), is a well-defined homomorphism \( \text{Aut}(\mathcal{H})/\mathcal{G} \longrightarrow \text{Aut}(\mathcal{X})/\mathcal{G} \) admitting a section and having kernel \( \text{Pic}(\mathcal{X})/\text{Pic}_0(\mathcal{X}) \). For the second sequence note, that for \( \phi \in \text{Aut}_0(\mathcal{H}) \) and any point \( x \) we have \( \phi \circ \sigma_x \circ \phi^{-1} = \sigma_x \), hence \( \phi \circ \sigma \circ \phi^{-1} = \sigma \), thus the map \( \phi \mapsto \sigma^n \circ \phi \) such that \( \sigma^n(\phi(L)) \simeq L \) is a homomorphism. The final assertion is clear by considering the map \( \text{Aut}(\mathcal{X})/\mathcal{G} \longrightarrow \text{Aut}(\mathcal{H})/\text{Aut}_0(\mathcal{H}) \), which is induced by the inclusion. \( \square \)
Proposition 5.2.3 will be extended to weighted curves in Proposition 6.3.4.

5.3. The twisted polynomial case

We now discuss the case, where the underlying bimodule is non-simple, moreover of the form $M(F, \alpha)$. That is, we assume that the derivation $\delta$ is trivial. In the special case where the automorphism $\alpha \in \text{Aut}(F/k)$ is a generator of the Galois group of the centre $Z/k$ we determine the automorphism group $\text{Aut}(X)$ completely. It follows in particular that $\alpha$ induces a generator of the ghost group. In the next section we will deduce a formula for the Auslander-Reiten translation.

5.3.1. Let $M = M(F, \alpha)$. Recall that $\Pi(L, \sigma) \simeq F[X; Y, \alpha]$, where $X$ is central. Let $x$ and $y$ be the points corresponding to the prime ideals generated by $X$ and $Y$, respectively, and let $\sigma_x$ and $\sigma_y$ be the associated tubular shifts.

Modulo inner automorphisms $\alpha$ has finite order $r$. There is some $u \in \text{Fix}(\alpha)^*\alpha$ such that $\alpha^r(f) = u^{-1}fu$ for all $f \in F$. Recall that for an element $f \in F$ the norm of $f$ is given by $N(f) = \alpha^{-1}(f) \cdots \alpha(f)f$.

Denote by $\text{Gal}(F/k)$ the factor group $\text{Aut}(F/k)/\text{Inn}(F/k)$. By the Skolem-Noether theorem [29], restriction to the centre $Z = Z(F)$ induces a monomorphism $\text{Gal}(F/k) \subset \text{Gal}(Z/k)$. Note that if $k$ is the field of real numbers or a finite field (hence $F$ commutative), then $\text{Gal}(F/k)$ is cyclic.

5.3.2. The automorphism $\alpha \in \text{Aut}(F/k)$ induces an automorphism of the bimodule $M$, given by $(f, g) \mapsto (\alpha(f), \alpha(g))$, which we denote also by $\alpha$. Denote by $\alpha_* \in \text{Aut}(X)$ the induced automorphism of the curve, as described in the proof of 5.1.4.

Let $\gamma$ be the graded algebra automorphism of $R = F[X; Y, \alpha]$ given by $rY = Y \gamma(r)$ for all $r \in R$. Then $\alpha_*$ coincides with $(\gamma^{-1})_* = \gamma^*$ (as defined in 3.2.1). In fact, it is easy to see that these automorphisms coincide on the full subcategory $\{L, \mathcal{L} = L(1)\}$, and by the argument given in the proof of Proposition 5.1.4 they coincide also on $\mathcal{H}$.

From the description of the prime elements in 1.7.7 the next lemma follows immediately:

**Lemma 5.3.3.** $\alpha$ induces a prime fixing automorphism of $R$. Accordingly $\alpha_*$ is a ghost of order $r$.

For the automorphism group we treat a special, cyclic Galois case.

**Theorem 5.3.4.** Let $Z$ be the centre of $F$ and $\alpha \in \text{Aut}(F/k)$ such that its image in $\text{Gal}(F/k)$ generates the group $\text{Gal}(Z/k)$ of order $r$.

1. Let $r = 1$. Then $\text{Aut}(X)$ is canonically isomorphic to $\text{PGL}_2(Z)$.
2. Let $r \geq 2$. Then $\text{Aut}(X)$ is generated by
• the automorphism $\alpha_\ast$ induced by $\alpha$, which generates the ghost group $\mathcal{G}$ and is of order $r$;
• transformations of the form $Y \mapsto aY$ for $a \in \mathbb{Z}^*$; they are liftable to $R$; two transformations $Y \mapsto aY$ and $Y \mapsto bY$ (with $a, b \in \mathbb{Z}^*$) give the same automorphism on $X$ if and only if $N(a) = N(b)$;
• if $r = 2$ additionally by the (non-liftable) automorphism exchanging $X$ and $Y$.

Proof. We assume $r \geq 2$, since the case $r = 1$ is easy. Let $M$ be the underlying bimodule as described above. By 5.1.4, $\text{Aut}(\mathbb{X})$ is given by $\text{Out}(M) = \text{Aut}(M)/\text{Inn}(M)$. Elements of $\text{Aut}(M)$ are given by triples $(\varphi_1, \varphi_M, \varphi_2)$ with $\varphi_1, \varphi_2 \in \text{Aut}(F/k)$ and $\varphi_M \in \text{End}_k(M)$ bijective such that $\varphi_M(f_1 m f_2) = \varphi_1(f_1) \varphi_M(m) \varphi_2(f_2)$ for all $f_1, f_2 \in F$ and $m \in M$. Note that modulo inner bimodule automorphisms we can assume that $\varphi_1$ and $\varphi_2$ are powers of $\alpha$. Recall that there is the automorphism induced by $(\alpha, \alpha, \alpha)$, where on $F M = F \oplus F$ it is given by $\alpha(x, y) = (\alpha(x), \alpha(y))$. By abuse of notation, we denote this automorphism of $M$ by the same letter $\alpha$.

We have $\alpha_\ast = 1$. Note that in case $\varphi_1 = 1$, $\varphi_M$ is given by an invertible matrix,

$$\varphi_M(x, y) = (x, y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d \in F$ such that $ad - bc \neq 0$. Calculating $\varphi_M((1, 0) \cdot f)$ and $\varphi_M((0, 1) \cdot f)$ for all $f \in F$, one gets in case $\varphi_2 = 1$ that the matrix is diagonal with entries lying in $\mathbb{Z}^*$ (exploit the existence of $f \in \mathbb{Z}$ such that $\alpha(f) \neq f$). Similarly, an automorphism of the form $(1, \varphi_M, \alpha^j)$ with $1 \leq j \leq r - 1$ is only possible for $r = 2$ and $j = 1$, which leads to the matrices of the form

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

with $a, b \in \mathbb{Z}^*$ (using $f \in \mathbb{Z}$ such that $\alpha^j(f) \neq f$). Moreover, a diagonal matrix is inner if and only if it is of the form

$$\begin{pmatrix} ab & 0 \\ 0 & a \alpha(b) \end{pmatrix}$$

in $\mathbb{Z}^*$. The norm $N$ induces a map, assigning a diagonal matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

the value $N(a^{-1} b)$, and, involving Hilbert’s Theorem 90 (applied to the cyclic Galois extension $\mathbb{Z}/K$ where $K = \mathbb{Z} \cap \text{Fix}(\alpha)$, see [62]), such a matrix is mapped to the identity if and only if it is of the above diagonal form with twist. Moreover, up to inner automorphisms we can assume $a = 1$, hence we have a transformation $Y \mapsto bY$. Since $(bY) f = \alpha(f)(bY)$ for all $f \in F$, this transformation extends to a graded $k$-algebra automorphism of $R = F[X; Y, \alpha]$, mapping $X^i Y^j$ to $N_j(b) X^i Y^j$. Moreover, for $N(b) \neq 1$ this induces a geometric element in $\text{Aut}(\mathbb{X})$: otherwise it would be prime fixing, in particular it would fix the prime element $X^r + u Y^r$, and then $N(b) = N_r(b) = 1$ would follow. □
Remark 5.3.5. (1) With the assumptions and notations of the theorem
let \( r \geq 2 \) and \( K = Z \cap \text{Fix}(\alpha) \). Let \( U \) be the subgroup of \( Z^* \) of elements \( a \) with \( N(a) = 1 \). Then
\[
\text{Aut}(X) \simeq \begin{cases} 
(Z^*/U \rtimes \mathbb{Z}_2) \rtimes \langle \alpha \rangle & r = 2; \\
Z^*/U \rtimes \langle \alpha \rangle & r > 2;
\end{cases}
\]
where \( \mathcal{G} = \langle \alpha \rangle \) is the cyclic ghost group of order \( r \). Let \( K^+ = N(Z^*) \subseteq K^* \).

Then \( N \) induces an isomorphism of groups \( Z^*/U \simeq K^+ \). But note that the action of \( K^+ \) on \( X \) is not given explicitly.

(2) In case \( r = 2 \), although \( X \) is central in \( R \) and \( Y \) is not, the localizations \( R_0^0(X) \) and \( R_0^0(Y) \) are isomorphic (by exchanging \( XY^{-1} \) and \( YX^{-1} \)).

(Compare 2.2.5.)

Examples 5.3.6. (1) The theorem can be applied for \( k = \mathbb{R} \) to the bimodules \( M = \mathbb{R} \oplus \mathbb{R} \) and \( M = \mathbb{C} \oplus \overline{\mathbb{C}} \), where \( \mathbb{C} \) acts on the right hand side on the second component via complex conjugation \( \alpha \). Then \( r = 2 \), and the elements of \( \text{Aut}(X) \) are given by \( Y \mapsto rY \) \((r > 0)\), by “inversion” \( X \leftrightarrow Y \) and by the ghost \( \alpha \).

(3) Let \( k = \mathbb{Q}(i) \) and \( F = k(\sqrt{2}) \), let \( \alpha \) be the automorphism \( \sqrt{2} \mapsto i\sqrt{2} \) and let \( M = M(F, \alpha) \). Then \( \text{Aut}(X) \) consists of the ghost \( \alpha \) of order 4 and the automorphisms \( Y \mapsto aY \) \((a \in F^*)\). Here an element \( a \in F^* \) cannot always be represented (modulo the group \( U \) of elements of norm 1) by an element in \( \text{Fix}(\alpha) = \mathbb{Q}(i) \). For example, \( N(\sqrt{2}) = -2 \) cannot coincide with the norm of an element in \( \mathbb{Q}(i)^* \).

5.4. On the Auslander-Reiten translation as functor

We continue to study the non-simple bimodule case where we still assume that for the derivation we have \( \delta = 0 \). Hence the orbit algebra is of the form \( F[X; Y, \alpha] \). As before, let \( x \) and \( y \) be the points corresponding to the prime elements \( X \) and \( Y \), respectively.

Theorem 5.4.1. Let \( R = \Pi(L, \sigma_y) = F[X; Y, \alpha] \) with \( \alpha \in \text{Aut}(F/k) \). Let \( r \) be the order of \( \alpha \) modulo inner automorphisms. Then

(1) As elements of \( \text{Aut} \mathcal{H}, \sigma_x^{-1} \circ \sigma_y = \alpha \).

(2) \( \text{Pic}(X) = \langle \sigma_x, \sigma_y \rangle \simeq \mathbb{Z} \times \mathbb{Z}_r \) and \( \text{Pic}_0(X) = \langle \alpha \rangle \simeq \mathbb{Z}_r \).

(3) The Auslander-Reiten translation \( \tau \) acts on elements of \( \text{End}(L) \) like \( \sigma_x^{-1} \circ \sigma_y^{-1} \).

Proof. (1) Let \( \gamma \) be the automorphism of the graded algebra \( R \) given by \( r \pi_y = \pi_y \gamma(r) \) (which coincides on \( R_0 \) with \( \alpha^{-1} \)). Then by definition, \( \gamma^r = \alpha \), and the formula follows from 3.2.8.
(2) As before, let \( u \in \text{Fix}(\alpha)^* \) such that \( \alpha^r(f) = u^{-1}fu \) for all \( f \in F \). Then, \( \pi_x \in R \) is central, \( \pi_y \) is normal such that \( u\pi_y \) is central, and every other prime element in \( R \) is central. Moreover by (1), for \( 1 \leq j \leq r - 1 \), the automorphisms \( \sigma_j^y \) differ from the powers of \( \sigma_x \). Hence \( \text{Pic}(X) = \langle \sigma_x, \sigma_y \rangle \simeq \mathbb{Z} \times \mathbb{Z}_r \) follows from Corollary 3.2.10. Moreover, \( \text{Pic}_0(X) \) is generated by \( \sigma_x^{-1} \circ \sigma_y = \alpha_* \).

(3) Let \( m \) be the homogeneous Jacobson radical of the graded local ring \( R \). From the diagram of the Koszul complex (see 2.1.8)

\[
\begin{array}{c}
0 & \longrightarrow & R(-2) & \longrightarrow & R(-1) \oplus R(-1) & \longrightarrow & R & \longrightarrow & R/m \\
\downarrow{g} & & \downarrow{a} & b & \longrightarrow & \downarrow{c} & d & \longrightarrow & \downarrow{f} \\
0 & \longrightarrow & R(-2) & \longrightarrow & R(-1) \oplus R(-1) & \longrightarrow & R & \longrightarrow & R/m
\end{array}
\]

(where we dropped the shifts for the maps) it follows that \( b = 0 = c, a = f, d = \alpha^{-1}(f)(-1), \) and \( g = \alpha^{-1}(f)(-2) \). By sheafification we get for any \( f \in \text{End}(L) \) a diagram of almost split sequences

\[
\mu : 0 \longrightarrow \tau L = L(-2) \longrightarrow L(-1) \oplus L(-1) \longrightarrow L \longrightarrow 0
\]

\[
\mu : 0 \longrightarrow \tau L = L(-2) \longrightarrow L(-1) \oplus L(-1) \longrightarrow L \longrightarrow 0.
\]

It follows that (on classes) \( \sigma_x^{-2} \alpha^{-1}(f) \cdot \mu = \mu \cdot f \), for any \( f \in \text{End}(L) \). On the other hand, as can be derived from [74, Lemma 3], \( \tau(f) \cdot \mu = \mu \cdot f \), and \( \tau(f) = \sigma_x^{-2} \alpha^{-1}(f) \) follows. Now apply (1). \( \square \)

Let \( S_x \) and \( S_y \) be the simple objects associated to \( \pi_x \) and \( \pi_y \), respectively. Mapping \( f \in \text{End}(L) \) to its fibre map \( f_x \) induces an isomorphism \( \text{End}(S_x) \simeq \text{End}(L) = F \), and similarly for \( S_y \). Then the following is easy to see.

**Corollary 5.4.2.** On elements of \( \text{End}(S_x) \) and \( \text{End}(S_y) \) the Auslander-Reiten translation \( \tau \) acts like \( \alpha^{-1} \in \text{Aut}(F/k) \). \( \square \)

**Corollary 5.4.3.** On the tube \( \mathcal{U}_x \) (\( \mathcal{U}_y \)) the tubular shift \( \sigma_x \) (\( \sigma_y \)) coincides with \( \alpha^* \) (\( \alpha_* \), respectively) and hence does not coincide, in case \( r \geq 2 \), with the identity functor on this tube. \( \square \)

**Corollary 5.4.4.** Assume, that the powers of \( \alpha_* \) are the only ghosts (which is true, for example, under the assumptions of Theorem 5.3.4). Then as elements in \( \text{Aut}(\mathcal{H}) \),

\[
\tau = \sigma_x^{-1} \circ \sigma_y^{-1} = \sigma_x^{-2} \circ \alpha^*.
\]
5.5. The quaternion case

Let $k$ be a field of characteristic different from two. Let $a, b \in k^*$ and let $F = (\frac{a,b}{k})$ be an algebra of quaternions over $k$, that is, a $k$-algebra on generators $i$ and $j$ subject to the relations $ji = -ij$, $i^2 = a, j^2 = b$. Moreover, we assume that $F$ is a skew field. Let $M$ be the bimodule $kF_F$. Let $F_0 = k_i \oplus k_j \oplus k_{ji}$ be the quadratic space of pure quaternions, where the quadratic form is given by the restricted norm form $q = -aX^2 - bY^2 + abZ^2$. Let $\text{SO}(F_0)$ be the group of all isometries of this quadratic space with determinant 1 (see [60]).

**Proposition 5.5.1.** Let $X$ be a homogeneous exceptional curve, where the underlying bimodule is given as above. We have an isomorphism $\text{Aut}(X) \cong \text{SO}(F_0)$ which induces the canonical action of $\text{SO}(F_0)$ on the projective spectrum of the coordinate algebra $R = k[X,Y,Z]/(-aX^2 - bY^2 + abZ^2)$. Each automorphism of $X$ is geometric.

**Proof.** We exhibit the proof which is given in [57] for $k = \mathbb{R}$. We have to calculate the (outer) automorphisms of the bimodule $M$. For each $a \in F^*$ denote by $\iota_a : F \rightarrow F$ the inner automorphism given by $\iota_a(f) = a^{-1}fa$ for all $f \in F$. Each $\varphi \in \text{Aut}(M)$ has the form $\varphi = (1, \varphi, \iota_a)$, where $\varphi(f) = \varphi(1)a^{-1}fa$. We obtain a surjection $F^* \times F^* \rightarrow \text{Aut}(M)$ with kernel $1 \times k^*$, hence $\text{Aut}(M) \simeq F^* \times F^*/k^*$. Since every inner automorphism of the bimodule $M$ is of the form $x \mapsto \alpha^{-1}xa$ for some $\alpha \in k^*$, $a \in F^*$, there is a surjection $k^* \times F^* \rightarrow \text{Im}(M)$ inducing an isomorphism $\text{Im}(F) \simeq k^* \times F^*/k^*$. Hence $\text{Out}(M) \simeq F^*/k^* \simeq \text{SO}(F_0)$ (see [60]). By the correspondence
between the basis $i, j, k = ji$ of $F_0$ and the elements $x, y, z$ in $R$ (as described in [53, 4.3]), we see how an element of $SO(F_0)$ acts on (prime) elements of degree one in $R$, and this action extends uniquely to an automorphism of the graded $k$-algebra $R$.

5.6. The homogeneous curves over the real numbers

In this section we apply and illustrate our results on the automorphism group in the special situations where $k = \mathbb{R}$ is the field of real numbers. There are (up to duality) only five tame bimodules, which are listed in the following table. The corresponding graded factorial coordinate algebras and automorphism groups $\text{Aut}(X)$ were determined in the preceding sections. In the table, $\gamma$ denotes complex conjugation, $I$ the inversion $z \mapsto 1/z$ (explained below). Note that in the “classical” case $\mathbb{C} \oplus \mathbb{C}$ complex conjugation occurs, since we consider it as bimodule over $\mathbb{R}$. Moreover, in this case complex conjugation induces a geometric automorphism.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$R = \Pi(L, \sigma_x)$</th>
<th>$\text{Aut}(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$\mathbb{R}[X,Y,Z]/(X^2 + Y^2 + Z^2)$</td>
<td>$\text{SO}_3(\mathbb{R})$</td>
</tr>
<tr>
<td>2.</td>
<td>$\mathbb{R}[X,Y]$</td>
<td>$\text{PGL}_2(\mathbb{R})$</td>
</tr>
<tr>
<td>3.</td>
<td>$\mathbb{C}[X,Y]$</td>
<td>$\text{PGL}_2(\mathbb{C}) \rtimes \langle \gamma \rangle$</td>
</tr>
<tr>
<td>4.</td>
<td>$\mathbb{H}[X,Y], X, Y$ central</td>
<td>$\text{PGL}_2(\mathbb{H})$</td>
</tr>
<tr>
<td>5.</td>
<td>$\mathbb{C}[X,Y], X$ central, $Yz = \overline{Y}$</td>
<td>$(\mathbb{R}_+ \rtimes \langle I \rangle) \rtimes \langle \gamma \rangle$</td>
</tr>
</tbody>
</table>

Table 5.1. The real homogeneous curves

5.6.1. Let $R = \Pi(L, \sigma)$, where $\sigma$ is an efficient automorphism. The points of $X$ are the prime elements in $R$. In each of the five cases we list the prime elements (up to units), the endomorphism skew fields of the corresponding simple objects $S_x$ and the symbol data $\left(\frac{d(x)}{f(x)}\right)$. We call a point $x \in X$ real (complex, quaternion) if the endomorphism ring of $S_x$ is $\mathbb{R}$ ($\mathbb{C}, \mathbb{H}$, respectively). We call the property of $x$ being real, complex or quaternion, respectively, also the colouring of $x$.

1. $\mathbb{R}[X,Y,Z]/(X^2 + Y^2 + Z^2) = \mathbb{R}[x,y,z]$.
   - $ax^2 + by^2 + cz^2$, $(a, b, c) \neq (0, 0, 0)$; $\mathbb{C}$; $1_1$

Hence $X$ can be identified with $\mathbb{S}^2/\pm 1$, the 2-sphere modulo antipodal points. This is homeomorphic to $\mathbb{P}^1(\mathbb{C})/\mathbb{Z}_2$, the Riemann sphere modulo the fixed-point free involution (given by $z \mapsto -1/\bar{z}$ on $\mathbb{P}^1(\mathbb{C})$). There are no real points.
5.6. THE HOMOGENEOUS CURVES OVER THE REAL NUMBERS

(2) \( \mathbb{R}[X, Y] \).

- \( X, Y + \alpha X \quad \alpha \in \mathbb{R}; \mathbb{R}; \left( \frac{1}{1} \right) \).
- \( (Y + zX)(Y + \bar{z}X) \quad z \in \mathbb{C} \setminus \mathbb{R}; \mathbb{C}; \left( \frac{2}{1} \right) \).

Hence \( X = \mathbb{P}^1(\mathbb{C})/Z_2 \) (identifying \( X, Y + \alpha X, (Y + zX)(Y + \bar{z}X) \) with the class of \( \infty, \alpha, z \), respectively) where here \( Z_2 \) is generated by the involution (given by \( z \mapsto \bar{z} \)) having fixed points (\( \sim \) real points).

We have two regions, the boundary (\( \sim \) real points) having symbol data \( \left( \frac{1}{1} \right) \) and the inner points are complex having symbol data \( \left( \frac{2}{2} \right) \).

(3) \( \mathbb{C}[X, Y] \).

- \( X, Y + zX \quad z \in \mathbb{C}; \mathbb{C}; \left( \frac{1}{1} \right) \).

Here, \( X = \mathbb{P}^1(\mathbb{C}) \), the Riemann sphere.

(4) \( \mathbb{H}[X, Y] \).

- \( X, Y + \alpha X \quad \alpha \in \mathbb{R}; \mathbb{H}; \left( \frac{1}{1} \right) \).
- \( (Y + zX)(Y + \bar{z}X) \quad z \in \mathbb{C} \setminus \mathbb{R}; \mathbb{C}; \left( \frac{2}{1} \right) \).

Here \( X = \mathbb{P}^1(\mathbb{C})/Z_2 \) (as in case (2)), but the boundary is coloured quaternion.

(5) \( \mathbb{C}[X, \overline{Y}] \).

- \( X, Y; \mathbb{C}; \left( \frac{1}{1} \right) \)
- \( Y^2 - \alpha X^2 = (Y - \sqrt{\alpha}X)(Y + \sqrt{\alpha}X) \quad 0 < \alpha \in \mathbb{R}; \mathbb{R}; \left( \frac{2}{1} \right) \)
- \( Y^2 - \alpha X^2 \quad 0 > \alpha \in \mathbb{R}; \mathbb{H}; \left( \frac{2}{1} \right) \)
- \( (Y^2 - zX^2)(Y^2 - \bar{z}X^2) \quad z \in \mathbb{C} \setminus \mathbb{R}; \mathbb{C}; \left( \frac{1}{2} \right) \).

In this case, the points of \( X \) are in one-to-one correspondence with the elements of \( \mathbb{P}^1(\mathbb{C})/Z_2 \) (mapping \( X, Y, Y^2 - \alpha X^2 \) (\( 0 \neq \alpha \in \mathbb{R} \)), \( (Y^2 - zX^2)(Y^2 - \bar{z}X^2) \) (\( z \in \mathbb{C} \setminus \mathbb{R} \)) to the class of \( \infty, 0, \alpha, z \) in \( \mathbb{P}^1(\mathbb{C})/Z_2 \), respectively). The boundary is coloured in a more complicated fashion as in the preceding cases and is indicated in Figure 5.1.

Thus, in each of the five cases \( X \) can be identified with the Riemann sphere \( \mathbb{P}^1(\mathbb{C}) \) or a quotient of it modulo some involution plus some additional structure. Non-boundary points are always complex. \( \square \)
Proposition 5.6.2 ([57]). Let $X$ be a homogeneous exceptional curve over $k = \mathbb{R}$. The geometric automorphism group of $X$ is canonically isomorphic to the group of directly conformal homeomorphisms on the Riemann sphere respecting the involution and the colouring.

Proof. The geometric elements of $\text{Aut}(X)$ for all real cases are explicitly described in 5.3.4 and 5.5.1 as certain invertible $2 \times 2$-matrices. Thus these elements act in a natural way as Möbius transformations on the Riemann sphere, which induces a surjective homomorphism from $\text{Aut}(X)$ to the group of conformal homeomorphisms respecting the involution and the colouring [45] (ghosts mapped to the identity), which establishes the isomorphism as claimed. Moreover, having identified $X$ with the Riemann sphere as above, this action coincides with the action of $\text{Aut}(X)$ on $X$, with the only exception that in case $C[X, Y]$ a transformation of the form $X \mapsto aX$ corresponds to the Möbius transformation $z \mapsto a^2z$. □

In case $k = \mathbb{R}$ Proposition 5.2.3 reads as follows.

Proposition 5.6.3. Let $X$ be a homogeneous exceptional curve over $k = \mathbb{R}$. Let $\mathcal{G}$ be the ghost group. Then the group $\text{Pic}(X)/\mathcal{G}$ acts simply transitive on the set of all (isomorphism classes of) line bundles and there is a split exact sequence of groups

$$1 \rightarrow \text{Pic}(X)/\mathcal{G} \rightarrow \text{Aut}(\mathcal{H})/\mathcal{G} \rightarrow \text{Aut}(X)/\mathcal{G} \rightarrow 1.$$ □

The weighted version is given in 6.3.5. We restate Corollary 5.4.4:

Proposition 5.6.4. Let $k = \mathbb{R}$. Only in case $M = \mathbb{C} \oplus \bar{\mathbb{C}}$ the ghost group $\mathcal{G}$ is non-trivial. In this case $\mathcal{G}$ is generated by the automorphism $\gamma$ of order two induced by complex conjugation. Let $x$ and $y$ be the unirational points. Then

$$\tau = \sigma_x^{-1} \circ \sigma_y^{-1} = \sigma_x^{-2} \circ \gamma.$$
5.7. Homogeneous curves with finite automorphism group

Over a finite field the automorphism group of a homogeneous curve $X$ is finite (by 5.1.4). But also in characteristic zero the automorphism group of a homogeneous curve may be finite.

**Example 5.7.1.** Let $F/k$ be commutative with $[F : k] = 4$ and $M = kF_F$. Then it is easy to see that $\text{Out}(M) \cong \text{Gal}(F/k)$.

**Example 5.7.2.** Let $M$ be the $\mathbb{Q}$-$\mathbb{Q}(\sqrt{2}, \sqrt{3})$-bimodule $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. With the notations as in 4.3.7, the elements $y$ and $z$ in $R = \Pi(L, \sigma_x)$ are not central up to a unit, hence define (non-trivial) ghosts of order 2 (by 3.2.3). In fact they generate $\text{Aut}(X)$ which is isomorphic to the Klein four group $V_4$. Each automorphism of $X$ (except the identity) is a ghost.

In the rest of this section we will elaborate a more complicated example which we will meet again later when discussing a tubular curve of index three, see 8.3.1.

**Example 5.7.3.** Let $k = \mathbb{Q}$ and $F = (\mathbb{Q}(-1,-1))$ be the skew field of quaternions over $\mathbb{Q}$ on generators $i, j$ with relations $i^2 = -1 = j^2$, $ij = -ji$, $K = \mathbb{Q}(\sqrt{-3}, \sqrt{2})$ and $M$ be the bimodule $K(K \oplus K)_F$ with the canonical $K$-action, and where the $F$-action on $M$ is defined by

$$(x, y) \cdot i = \frac{1}{\sqrt{-3}}(\sqrt{2}x + y, x - \sqrt{2}y), \ (x, y) \cdot j = (y, -x)$$

for all $x, y \in K$. Since $K \not\cong F$, the bimodule $M$ is simple. Let $X$ be the homogeneous exceptional curve over this bimodule.

**Proposition 5.7.4.** $\text{Aut}(X) \cong V_4$, the Klein four group, and every automorphism is geometric. Moreover, there is a rational point $x$, which is fixed by all automorphisms.

**Proof.** We have to calculate the (outer) automorphisms $(\gamma, \varphi, \delta)$ of the bimodule $M$. Since every $\mathbb{Q}$-automorphism of $F$ is inner, modulo inner bimodule automorphisms we can assume $\delta = 1_F$. Moreover, for $\gamma$ we have the possibilities, that $\gamma$ is the identity, or $\gamma = \alpha$, $\beta$ or $\beta \alpha$, where $\text{Gal}(K/\mathbb{Q}) = \langle \alpha, \beta \rangle$, with

$$\begin{align*}
\alpha(\sqrt{2}) &= \sqrt{2} & \beta(\sqrt{2}) &= -\sqrt{2} \\
\alpha(\sqrt{-3}) &= -\sqrt{-3} & \beta(\sqrt{-3}) &= \sqrt{-3}.
\end{align*}$$

For $\gamma = 1_K$, using that $\varphi((1,0)i) = \varphi(1,0)i$ and $\varphi((1,0)j) = \varphi(1,0)j$ one immediately gets, that $\varphi$ is represented by a scalar matrix with non-zero entries in $K$, and hence induces the identity modulo inner automorphisms.
It is easy to see, that modulo inner, the only automorphism with $\gamma = \alpha$ is $(\alpha, \tilde{\alpha}, 1)$, where 
$$
\tilde{\alpha}(x, y) = (\alpha(x), \alpha(y)) \cdot j,
$$
which is not inner since $\alpha \neq 1$. Namely, if $(\alpha, \varphi, 1)$ is an additional auto-
morphism, then $(\alpha, \tilde{\alpha}, 1) \circ (\alpha, \varphi, 1) = (1, \tilde{\alpha}\varphi, 1)$. Moreover, $(\beta, \tilde{\beta}, 1)$ is an
automorphism, where 
$$
\tilde{\beta}(x, y) = \frac{\sqrt{2}}{\sqrt{-3}} \cdot (\beta(x), \beta(y)) \cdot j + \frac{1}{\sqrt{-3}} \cdot (\beta(x), \beta(y)),
$$
and this is modulo inner the only automorphism with $\gamma = \beta$. Composing
these two automorphisms gives modulo inner the unique automorphism with $\gamma = \beta\alpha$, and we get the Klein four group.

We have to show that each automorphism is geometric: For example,
we have $\tilde{\alpha}(\sqrt{2}, \sqrt{-3}) = -(\sqrt{-3}, \sqrt{2})$, and a simple calculation shows, that
there are no $a \in K^*, f \in F^*$ such that $-(\sqrt{-3}, \sqrt{2}) = a(\sqrt{2}, \sqrt{-3})f$. Hence $\tilde{\alpha}$ moves the point associated to $(\sqrt{2}, \sqrt{-3})$.

Let $m = (1, 1)$. Then $\tilde{\alpha}(m) = mj$ and $\tilde{\beta}(m) = mi$, hence $\tilde{\alpha}$ and $\tilde{\beta}$ fix the
point associated to $m$ (compare Lemma 0.6.3). \hfill $\square$

**Proposition 5.7.5.** Let $X$ be as above. Each rational point has multi-
plicity 2 or 4, and there are rational points in both cases. In particular, there
exists an efficient tubular shift $\sigma_x$.

**Proof.** By Lemma 0.6.1 the multiplicity of any rational point is 2 or
4. Let $m = (\sqrt{2} + \sqrt{-3} + \sqrt{-6}, 1) \in M$. Then the intersection $Km \cap mF$
is of dimension two over $Q$. (In fact, one calculates, that the elements $(a - \frac{1}{2}b\sqrt{-6}) \cdot m$ (with $a, b \in Q$) lie in the intersection.) By Lemma 0.6.1 the
corresponding point $x$ has multiplicity 2, and then the associated shift $\sigma_x$ is
efficient. Similarly, for $m = (1, 1)$ the intersection $Km \cap mF$ is of dimension
one over $Q$, hence the multiplicity of the corresponding point is 4. \hfill $\square$
Part 2

The weighted case
CHAPTER 6

Insertion of weights

In this chapter we show how results for homogeneous curves can be extended to weighted curves. We concentrate here on the graded factoriality and on the automorphism groups. As a technical tool we will make use of the \( p \)-cycle construction by Lenzing.

6.1. \( p \)-cycles

In this section we briefly describe the \( p \)-cycle construction by Lenzing \([67]\). It is related to the concept of a parabolic structure \([101]\). It follows from the construction that each exceptional curve arises from a \emph{homogeneous} exceptional curve by insertion of weights into a finite number of points. This is, together with its inverse process, the perpendicular calculus \([34]\), the most important tool in order to reduce problems to the homogeneous (=unweighted) case. We will consider the following problems:

- construction of graded factorial coordinate algebras for \( \mathcal{H} \) in the weighted cases by describing an equivalent process of insertion of weights into prime elements of the graded factorial algebras in underlying homogeneous case (see 6.2.4).
- determination of the automorphism group \( \text{Aut}(X) \) and the Picard group \( \text{Pic}(X) \) in the weighted cases (see 6.3.1 and 6.3.3).

6.1.1 \( (p \text{-cycles in } x) \). Let \( X \) be an exceptional curve with associated hereditary category \( \mathcal{H} \). Let \( x \in X \) be a homogeneous point and \( p \geq 2 \) be an integer ("weight"). We use the natural transformation \( 1_\mathcal{H} \xrightarrow{x} \sigma_x \). A \( p \)-cycle (concentrated) in \( x \) is an integer indexed sequence

\[
E = (E_i, x_i)_{i \in \mathbb{Z}} = [ \cdots \xrightarrow{x_0} E_0 \xrightarrow{x_1} E_1 \xrightarrow{x_2} E_2 \rightarrow \cdots \xrightarrow{x_{p-1}} E_{p-1} \xrightarrow{x_0} E_0(x) \rightarrow \cdots ]
\]

of morphisms \( E_i \xrightarrow{x_i} E_{i+1} \) in \( \mathcal{H} \) such that \( E_{i+p} = E_i(x) \) and \( x_{i+p} = \sigma_x(x_i) \) for all \( i \in \mathbb{Z} \) and each composition \( x_{i+p-1} \circ x_{i+p-2} \circ \cdots \circ x_i \) coincides with the natural morphism \( x_{E_i} \) for all \( i \in \mathbb{Z} \). We write

\[
E = [ E_0 \xrightarrow{x_0} E_1 \xrightarrow{x_1} E_2 \rightarrow \cdots \xrightarrow{x_{p-1}} E_{p-1} \xrightarrow{x_0} E_0(x) ].
\]

A morphism \( f \) between \( p \)-cycles \( E = (E_i, x_i) \) and \( F = (F_i, y_i) \) in \( x \) consists of morphisms \( f_i : E_i \rightarrow F_i \) such that \( f_{i+1} \circ x_i = y_i \circ f_i \) and \( f_{i+p} = f_i(x) \) for all \( i \in \mathbb{Z} \).
The category of all \( p \)-cycles in \( x \) is denoted by \( \mathcal{H} = \mathcal{H}\left(\frac{p}{x}\right) \) and is easily seen to be abelian and noetherian.

6.1.2 (Inclusion). Let \( \mathcal{H} = \mathcal{H}\left(\frac{p}{x}\right) \). There is the full exact embedding \( j : \mathcal{H} \rightarrow \mathcal{H} \) given by

\[
j(E) = [E = E = E = \cdots = E \xrightarrow{x_{E}} E(x)]
\]

We have a left adjoint \( \ell : \mathcal{H} \rightarrow \mathcal{H} \) and a right adjoint \( r : \mathcal{H} \rightarrow \mathcal{H} \) of \( j \) given by \( \ell(E_i, x_i) = E_{p-1} \) and \( r(E_i, x_i) = E_0 \).

6.1.3 (Simple objects). The simple objects in \( \mathcal{H} \) are given (via the inclusion \( j \)) by the simple objects in \( \mathcal{H} \) concentrated in points \( y \) different from \( x \), together with the following \( p \)-cycles in \( x \):

\[
S_1 = [0 \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow S \rightarrow 0]
\]
\[
S_2 = [0 \rightarrow 0 \rightarrow \cdots \rightarrow S \rightarrow 0 \rightarrow 0]
\]
\[
\cdots
\]
\[
S_{p-1} = [0 \rightarrow S \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0]
\]
\[
S_p = [S \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \rightarrow S(x)]
\]

where \( S \in \mathcal{H} \) is simple and concentrated in \( x \). The simple objects \( S_1, \ldots, S_p \) are exceptional with \( \text{End}(S_i) = \text{End}(S) \) and \( \text{Ext}^1(S_{i+1}, S_i) \neq 0 \).

Let \( L \in \mathcal{H} \) be a special line bundle. Then \( j(L) \in \mathcal{H} \) is also special, since \( \text{Hom}(j(L), S_i) \neq 0 \) if and only if \( i = p \).

6.1.4 (Reduction of weights/perpendicular calculus). Let \( \mathcal{S} \) be the subset \( \{S_2, \ldots, S_p\} \) of the simple objects concentrated in \( x \) except \( S_1 \), and denote by \( \langle S \rangle \) the extension closure of \( \mathcal{S} \) in \( \mathcal{H} \). This forms a localizing Serre subcategory in \( \mathcal{H} \), and the functor \( \ell : \mathcal{H} \rightarrow \mathcal{H} \) induces an equivalence between the quotient category \( \mathcal{H}/\langle S \rangle \) and \( \mathcal{H} \).

The right perpendicular category \( \mathcal{S}^\perp \) formed in \( \mathcal{H} \) is equivalent to \( \mathcal{H} \).

6.1.5 (Tubular shifts). Let \( \mathcal{H} \) be the category of \( p \)-cycles in \( x \).

1. On \( \mathcal{H} \) the tubular shift \( \sigma_x \) associated to \( x \) corresponds to the natural transformation \( 1_{\mathcal{H}} \rightarrow \sigma_x \), indicated by the following diagram

\[
\begin{array}{cccccccc}
\mathcal{E} = [ & E_0 \xrightarrow{x_0} E_1 \rightarrow \cdots \rightarrow E_{p-2} \xrightarrow{x_{p-2}} E_{p-1} \xrightarrow{x_{p-1}} E_0(x) ] \\
\downarrow x_0 \quad \downarrow x_1 \quad \downarrow x_{p-2} \quad \downarrow x_{p-1} \quad \downarrow x_0(x) \\
\sigma_x \mathcal{E} = [ & E_1 \xrightarrow{x_1} E_2 \rightarrow \cdots \rightarrow E_{p-1} \xrightarrow{x_{p-1}} E_0(x) \xrightarrow{x_0(x)} E_1(x) ]
\end{array}
\]
(2) For every point \( y \in X \) the associated tubular shift \( \sigma_y : \mathcal{H} \to \mathcal{H} \) extends in an obvious way componentwise to an automorphism on \( \mathcal{H} \), again denoted by \( \sigma_y \), and which is for \( y \neq x \) just the tubular shift on \( \mathcal{H} \) associated to \( y \in X \). It is clear that the relations in \( \text{Aut}(\mathcal{H}) \) between the \( \sigma_y \, (y \in X) \) are the same as in \( \text{Aut}(\mathcal{H}) \). Moreover, \((\sigma_x)^p = \sigma_x\).

(3) More generally, each automorphism on \( \mathcal{H} \) which is point fixing can be extended componentwise onto \( \mathcal{H} \) (compare 0.4.8). In particular this holds for an efficient automorphism \( \sigma \) defined on the sheaf category over the underlying homogeneous curve.

6.1.6 (Reduction to the homogeneous case). We say that \( H(p_1, \ldots, p_t, x_1, \ldots, x_t) \) (which is defined inductively for pairwise different points \( x_1, \ldots, x_t \)) arises by insertion of a finite number of weights from \( H \). Let \( H \) be some abelian \( k \)-category. The following are equivalent [67, Thm. 5.1]:

1. \( H \) is equivalent to the hereditary category associated to an exceptional curve \( X \).
2. \( H \) arises by insertion of a finite number of weights from the hereditary category associated to a homogeneous exceptional curve.

6.2. Insertion of weights into central primes

We discuss the notion of insertion of weights on the level of the coordinate algebra. We will extend Serre’s theorem to weighted situations, at least in special cases. Generally it is not difficult to construct projective coordinate algebras in the weighted cases, even if there exists no tilting object. In the case of an exceptional curve, with the theorem by Artin and Zhang [2, Thm. 4.5] and by using a tilting bundle \( T \), forming an orbit algebra with respect to \( T \), the construction of a projective coordinate algebra is easy.

Proposition 6.2.1. Let \( X \) be an arbitrary exceptional curve, and let \( T \in H^+ \) be a tilting bundle. Let \( \sigma_x \) be the tubular shift at some point \( x \in X \). Then the pair \((T, \sigma_x)\) is ample. Hence \( \Pi(T, \sigma_x) \) is a projective coordinate algebra for \( X \).

Proof. The tilting bundle \( T \) defines the torsion class \( T = \{ X \in H | \text{Ext}^1(T, X) = 0 \} \). By Serre duality, \( H_0 \subset T \). Thus, if \( F \in H \), then by 0.4.6 and Serre duality we have \( F(n) \in T \) for sufficiently large \( n \). Then \( F(n) \) is a quotient of \( T^s \) for some \( s > 0 \) (see [39]). This implies the first property of ampleness, the second follows again with 0.4.6 and Serre duality.

A different and more general construction of an ample pair is described in [86, IV.4] (and [85, Appendix A]), which leads in our setting to an ample pair \((L, \sigma)\), where \( L \) is a (special) line bundle and \( \sigma \) is a composition of certain tubular shifts.
The projective coordinate algebras constructed in either of these two ways are not practical for our considerations. We would like to preserve the graded factoriality, starting from the homogeneous case.

6.2.2 (Insertion of weights into central primes). Let $X$ be a homogeneous exceptional curve with sheaf category $\mathcal{H}$ and $R = \Pi(L, \sigma)$ with $\sigma$ efficient, which is (positively) graded by the abelian group $H (= \mathbb{Z})$. Let $P = R\pi$ be a homogeneous prime ideal of height one, where we assume that $\pi$ is central and of degree $d$. (More generally, $R$ may be $k$-algebra, graded by an abelian group $H$, and $\pi$ a central, homogeneous element.) Let $p > 1$ be a “weight”.

Define $\overline{R} = R[\pi^{1/p}] = R[T]/(T^p - \pi)$, where $T$ is a central variable. Denote $\overline{H} = H[\overline{d}]$ and by $\overline{\pi} \in \overline{R}$ the image of $\pi$. Then $\overline{R}$ is an $\overline{H}$-graded algebra with $\deg(\pi) = \overline{h} = \frac{d}{p}$. If moreover $(H, \leq)$ is an ordered abelian group (defined by the positive cone $H_+ = \{h \in H \mid h \geq 0\}$) such that $R$ is positively $H$-graded ($R_h \neq 0$ only for $h \geq 0$), then $\overline{R}$ is positively $\overline{H}$-graded, where the ordering on $\overline{H}$ is defined by the positive cone $\overline{H}_+ = \{h + n\overline{h} \mid h \in H_+, n \geq 0\}$. This can be iterated.

Example 6.2.3. Let $p = (p_1, \ldots, p_t)$ and $d = (d_1, \ldots, d_t)$ be sequences of positive integers. Then the group $\mathbb{Z}[\frac{d_1}{p_1}, \ldots, \frac{d_t}{p_t}]$ is denoted by $\mathbb{L}(p, d)$. This is the abelian group given by generators $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_t$ and relations

$$p_i \bar{x}_i = d_i \bar{x}_0, \quad i = 1, \ldots, t.$$ 

If all $d_i = 1$, we write $\mathbb{L}(p)$ instead. Note that these groups can have a non-trivial torsion part.

Let $R = k[X_1, X_2]$ be the polynomial algebra graded by total degree. Let $\lambda_1, \ldots, \lambda_t$ be pairwise different elements in $k \cup \{\infty\}$. Without loss of generality we assume $\lambda_1 = \infty$ and $\lambda_2 = 0$. Let $\pi_i \in R$ be the homogeneous prime element $X_2 + \lambda_i X_1$ for $i = 3, \ldots, t$ and $\pi_1 = X_1$, $\pi_2 = X_2$. (In this case, all $d_i = 1$.) Successive insertion of weights $p_i > 1$ into the primes $\pi_i$ leads to the $\mathbb{L}(p)$-graded algebra

$$k[X_1, X_2, \ldots, X_t]/(X_i^{p_i} - X_2^{p_2} - \lambda_i X_1^{p_1} \mid i = 3, \ldots, t),$$

which are just the projective coordinate algebras of the weighted projective lines described in [33].

Theorem 6.2.4. Let $\pi \in R$ be a central prime element and $x \in X$ be the associated point. Let $p \geq 2$ be an integer.

(1) $\overline{R} = R[\pi^{1/p}]$ is an $\overline{H}$-graded factorial domain of Krull dimension two. More precisely, the homogeneous prime ideals in $\overline{R}$ of height one are $P = \overline{R}\pi$ and $P = \overline{R}q$, where $q \in R$ is prime and not associated to $\pi$.

(2) There is an equivalence $\frac{\text{mod}\, \frac{\mathbb{L}(\pi)}{\mathbb{L}(\pi)}}{\text{mod}\, \frac{\mathbb{L}(x)}{\mathbb{L}(x)}} \simeq \mathcal{H}\left(\frac{P}{x}\right)$. 
6.2. INSERTION OF WEIGHTS INTO CENTRAL PRIMES

Proof. (1) We have an embedding $\tilde{R} \subset R$, and $R$ can be considered also as $\overline{T}$-graded algebra. $\tilde{R} = R[\pi]$ is a finite centralizing extension, since $\overline{R} = R \oplus R\pi \oplus \cdots \oplus R\pi^{p-1}$. Hence, the intersection of a homogeneous prime ideal in $\overline{R}$ with $R$ gives a homogeneous prime ideal in $R$, proper inclusion is preserved, and each homogeneous prime ideal in $R$ is of this form (see [76, 10.]). Consequently, $\overline{R}$ is of graded Krull dimension two.

By the definition of the grading, every homogeneous element $\bar{\pi} \in \overline{R}$ has the form $\bar{\pi} = a\pi^l$, with $a \in R$ homogeneous and $0 \leq l \leq p - 1$. Hence, $\overline{R}$ is a graded domain like $R$.

It is easy to see that $\overline{\pi R} \cap R = \pi R$, hence there is an isomorphism $R/\pi R \simeq \overline{R}/\overline{\pi R}$, and it follows, that $\overline{\pi}$ is a central prime element in $\overline{R}$.

By the form of the homogeneous elements it follows easily that for a homogeneous prime ideal $Rq \subset R$ different from $R\pi$, the ideal $\overline{R}q$ is prime in $\overline{R}$. Moreover, by the surjection $P \rightarrow P \cap R$ as mentioned above, we see that every homogeneous prime ideal of height one in $\overline{R}$ different from $\overline{R}\overline{\pi}$ is of the form $Rq$, where $q$ is prime in $R$ and not associated to $\pi$.

(2) Denote by $\overline{H}$ the category of $p$-cycles concentrated in $x$. Let $\overline{\sigma}$ and $\Gamma_+$ be the functors as defined in 2.1.5. Extending this, we construct an exact functor $\overline{\sigma} : \text{mod}^H(\overline{R}) \rightarrow \overline{H}$ with kernel $\text{mod}^H_0(\overline{R})$. Denote by $r : \text{mod}^H(\overline{R}) \rightarrow \text{mod}^H(R)$ the exact functor, given by restricting an $\overline{H}$-graded module to the subgroup $H$. Obviously, $r(\overline{R}) = R$. Moreover, $M$ is of finite length over $\overline{R}$ if and only if $r(M(i\overline{h}))$ is of finite length over $R$ for all $i = 0, \ldots, p - 1$. Hence $r$ induces a functor $\overline{\pi}$ on the quotient categories. For $M \in \text{mod}^H(\overline{R})$ and $i = 0, \ldots, p$ define

\[
E_i = r(M(i\overline{h})) \in H.
\]

Then $E_p = E_0(x)$ and

\[
\overline{M} \defeq [ E_0 \xrightarrow{\overline{\pi}} E_1 \xrightarrow{\overline{\pi}} E_2 \rightarrow \cdots \rightarrow E_{p-1} \xrightarrow{\overline{\pi}} E_0(x) ]
\]

defines a $p$-cycle concentrated in $x$. In fact, by Theorem 3.1.2 multiplication $(\overline{\pi})^p = \cdot \pi$ induces the natural transformation $1 \xrightarrow{\cdot \pi} \sigma_x$. In this way we get the functor $\overline{\sigma}$ with the desired properties. By using $\overline{\pi}$ the induced functor on the quotient category is full, since the sheafification functor is.

It remains to show that $\overline{\sigma}$ is dense. Let

\[
\overline{E} = [ E_0 \xrightarrow{x_0} E_1 \xrightarrow{x_1} E_2 \rightarrow \cdots \rightarrow E_{p-1} \xrightarrow{x_{p-1}} E_0(x) ]
\]

be an arbitrary $p$-cycle concentrated in $x$. For each $i$, define $M_i = \Gamma_+ (E_i) \in \text{mod}^H(R)$ and $f_i = \Gamma_+ (x_i)$. Define an $\overline{R}$-module by $M = (M_0, \ldots, M_{p-1})$, where the homogeneous elements from $(M_i)_g$ have degrees in $g + i\overline{h}$ (for $g \in H$), and by defining the action of $\overline{\pi}$ on $M_i$ by $m \cdot \overline{\pi} = f_i(m)$. This is
well-defined, since multiplication with $\pi^p = \pi$ agrees with $f_{i+p-1} \circ \cdots \circ f_i$ (compare 3.1.1). Obviously, $\tilde{M} \simeq E$. □

**Remark 6.2.5.** (1) The proof of the denseness of $\tilde{\cdot} : \text{mod}^{H_0}(\tilde{R}) \to \mathcal{H}$ shows, that there is a functor $\Gamma_+ : \mathcal{H} \to \text{mod}^{H_+}(\tilde{R})$ such that $\tilde{\cdot} \circ \Gamma_+$ is isomorphic to the identity functor. Moreover, (restricting to positive gradings) it is right adjoint to $\tilde{\cdot}$, which follows from the adjointness of the corresponding functors $\tilde{\cdot} : \text{mod}^{H_+}(R) \to \mathcal{H}$ and $\Gamma_+ : \mathcal{H} \to \text{mod}^{H_+}(R)$ (compare 2.1.5).

(2) Degree shift by $\bar{h}$ on $\text{mod}^{H_0}(\tilde{R})$ corresponds to the tubular shift $\sigma_x$ on $\mathcal{H}$.

(3) Let $y \in X, y \neq x$ be a (homogeneous) point such that the corresponding prime $\pi_y$ is central. The conclusion in Theorem 3.1.2 (for $y$) extends (componentwise) to $\mathcal{H} = \mathcal{H}(\bar{p} \ x)$, that is, right multiplication with $\pi_y \in R \subset \tilde{R}$ induces the natural transformation $1_{\mathcal{H}} \to \sigma_y$. □

**Corollary 6.2.6.** Let $\pi_1, \ldots, \pi_t \in R$ be central prime elements, which are pairwise non-associated, let $x_1, \ldots, x_t \in X$ be the associated points, respectively. Denote by $\tilde{R}$ the algebra obtained from $R$ by insertion of weights $p_1, \ldots, p_t \geq 2$ into $\pi_1, \ldots, \pi_t$, respectively. Then $\tilde{R}$ is a graded factorial algebra and

$$\text{mod}^{\tilde{R}}(\tilde{R}) \simeq \mathcal{H}(\bar{p}_1, \ldots, p_t, x_1, \ldots, x_t).$$

**Proof.** With the preceding remark, we can apply the theorem inductively. □

**Remark 6.2.7.** Let $\tilde{R} = R[\pi^{1/p}]$ with $\pi \in R$ a central prime associated to a homogeneous point $x$.

(1) For $M = \tilde{R}$, the $E_i$ in (6.2.1) become $E_i = r(\tilde{R}_{\pi^i})$ ($i = 0, \ldots, p - 1$), so that $\tilde{\cdot}$ defines an isomorphism between $E_i$ and $E_{i+1}$ (for $i < p - 1$). Moreover, $E_p = R_0(h)$ and multiplication with $\tilde{\pi}$ gives only a monomorphism $E_{p-1} \to E_p$. We see, that the associated $p$-cycle $\tilde{L}$ is isomorphic to

$$j(L) = [ L = L = L = \cdots = L \xrightarrow{\pi} L(x) ].$$
where \( j : \mathcal{H} \longrightarrow \overline{\mathcal{H}} \) is the canonical embedding. Moreover, with the tubular shift \( \overline{\sigma_x} \) on \( \overline{\mathcal{H}} \) we have a morphism of \( p \)-cycles

\[
\overline{\mathcal{L}} = \begin{bmatrix}
L \rightarrow \bar{\mathcal{L}}
\end{bmatrix}
\]

\[
\overline{\sigma_x \mathcal{L}} = \begin{bmatrix}
L \rightarrow \bar{\mathcal{L}}
\end{bmatrix}
\]

It is easy to see that \( j(\mathcal{L}) \) is a special line bundle, if \( \mathcal{L} \) is.

(2) Assume that \( \pi = u_1 \ldots u_e \) with irreducible \( u_i \in R \) of degree \( f \). Then the morphism \( \overline{\mathcal{L}} \longrightarrow \overline{\sigma_x \mathcal{L}} \) from (1) factorizes into \( e \) morphisms between line bundles. For example, for \( e = 3 \) we get the following picture:

\[
\overline{\mathcal{L}} = \begin{bmatrix}
L \rightarrow \bar{\mathcal{L}}
\end{bmatrix}
\]

\[
\overline{\sigma_x \mathcal{L}} = \begin{bmatrix}
L \rightarrow \bar{\mathcal{L}}
\end{bmatrix}
\]

Note that multiplications with the (non-central) \( u_i \) from the left act on \( R \) as morphisms of right \( R \)-modules.

The line bundles lying in between \( \overline{\mathcal{L}} \) and \( \overline{\sigma_x \mathcal{L}} \) are not of the form \( \overline{\mathcal{R}}(g) \) for some \( g \in \overline{\mathcal{H}} \). (See also the example 8.5.5.)

(3) The restriction functor \( r : \text{mod} \overline{\mathcal{H}}(R) \longrightarrow \text{mod}^H(R) \), or more precisely, the induced functor \( \overline{r} \) between the quotient categories, plays the same role as the functor \( r : \overline{\mathcal{H}} \longrightarrow \mathcal{H} \) defined in 6.1.2.

With the results of Section 4.3 we get the following.

**Corollary 6.2.8.** Let \( X \) be an exceptional curve. The following statements are equivalent:

1. \( X \) is commutative.
2. \( X \) is multiplicity free.
3. \( X \) admits a commutative graded factorial domain as projective coordinate algebra.

As a consequence of this section, insertion of weights can be perfectly described on the level of the projective coordinate algebra as far as central
prime elements are concerned. One should emphasize that many interesting weighted examples can be examined already in this restricted context. Whereas insertion of weights is established very general by the \( p \)-cycle construction, a corresponding description on the level of the graded algebra for weight-insertion into non-central prime elements is still not available.

\textbf{Problem 6.2.9.} Describe the concept of insertion of weights into prime elements which are not central.

\section*{6.3. Automorphism groups for weighted curves}

Each exceptional curve arises by insertion of weights at finitely many points for some homogeneous exceptional curve \([67]\). The following proposition reduces the problem of calculating the automorphism group essentially to the homogeneous case.

Let \( X \) be a (homogeneous) exceptional curve and \( \phi \in \text{Aut}(X) \). Recall that \( \overline{\phi} \) denotes the shadow of \( \phi \). Let \( p : X \to \mathbb{N} \) be a weight function. Then \( \phi \) is called \textit{weight preserving} (with respect to \( p \)), if \( p(\overline{\phi}(x)) = p(x) \) for all \( x \in X \). We have in mind the following situation: Let \( x_1, \ldots, x_t \in X \) be distinguished points and \( p_1, \ldots, p_t \) weights. Then let \( p \) be defined by \( p(x_i) = p_i \) \( (i = 1, \ldots, t) \) and \( p(x) = 1 \) for all \( x \) different from the points \( x_1, \ldots, x_t \).

\textbf{Proposition 6.3.1.} Let \( \overline{X} \) be an exceptional curve with underlying homogeneous exceptional curve \( X \) such that \( \overline{X} \) arises from \( X \) by insertion of be weights \( p_1, \ldots, p_t \) into the distinct points \( x_1, \ldots, x_t \), respectively. Then \( \text{Aut}(\overline{X}) \) can be identified with the subgroup of elements in \( \text{Aut}(X) \) which preserve the weights.

\textbf{Proof.} Let \( u \in \text{Aut}(\overline{X}) \). Then there is a unique \( v \in \text{Aut}(\mathcal{H}) \) such that \( \ell u = \nu \ell \), where \( \ell : \overline{\mathcal{H}} \to \mathcal{H} \) is left adjoint to the inclusion \( j : \mathcal{H} \to \overline{\mathcal{H}} \), compare 6.1.2. Moreover, \( \ell j \simeq 1_{\mathcal{H}} \). Let \( L \) be the structure sheaf of \( \overline{X} \) as before. Then \( jL \) is a special line bundle of \( \overline{X} \). Since \( u(jL) = jL \), we get \( v(L) \simeq \nu(\ell jL) = \ell u(jL) = \ell j(L) \simeq L \). If \( u \simeq 1_{\mathcal{H}} \), then \( v \simeq \nu \ell j \simeq \ell u j \simeq \ell j \simeq 1_{\mathcal{H}} \). It follows that \( u \mapsto v \) defines a map \( \iota : \text{Aut}(\overline{X}) \to \text{Aut}(X) \), which is a homomorphism of groups.

We show injectivity of this map. Assume, that \( v \simeq 1_{\mathcal{H}} \). Then \( \ell u \simeq \ell \).

By 6.1.4 it follows, that \( u \) preserves all simple objects, hence \( u \circ \sigma_x \simeq \sigma_x \circ u \) for each \( x \in \overline{X} \) by 0.4.8. It follows, that \( u \) acts like the identity on the components of \( p \)-cycles in \( x \) and also on the components of morphisms between such cycles. Considering the natural transformation \( 1 \to \sigma_x \) it follows that \( u \) also acts as the identity on “horizontal” arrows in each cycle. Thus, \( u \) acts naturally as identity on \( p \)-cycles concentrated in \( x \), that is, \( u \simeq 1_{\overline{\mathcal{H}}} \).
Each \( v \in \text{Aut}(\mathcal{X}) \) lying in the image of \( \iota \) preserves the weights. Conversely, assume that \( v \in \text{Aut}(\mathcal{X}) \) preserves the weights \( p_1, \ldots, p_t \). Then \( v \) can be extended “componentwise” onto cycles, and inductively to an element \( u \in \text{Aut}(\mathcal{X}) \) such that \( \ell u = \nu \ell \). Hence \( v = \iota(u) \). (For example, if \( v(x_1) = x_2 \) and \( p_1 = p(x_1) = p(x_2) = p_2 \), then \( v \) can be extended to \( u : H\left(\frac{p_1}{x_1}\right) \to H\left(\frac{p_2}{x_2}\right) \), and this can be continued.)

**Corollary 6.3.2.** Let \( \mathcal{X} \) be an exceptional curve. Then the isomorphism class of \( \text{Aut}(\mathcal{X}) \) is independent of the chosen special line bundle \( L \) as structure sheaf.

**Proof.** Let \( L \) and \( L' \) be two special line bundles. After applying suitable tubular shifts associated to exceptional points, \( L \) and \( L' \) are special with respect to the same set of exceptional simple objects. By perpendicular calculus \( L \) and \( L' \) correspond to line bundles on the underlying homogeneous curve \( \mathcal{Y} \). But the definition of \( \text{Aut}(\mathcal{Y}) \) is clearly independent of the choice of the structure sheaf. Then the assertion follows by observing that if an automorphism \( \phi \) of \( \mathcal{H} \) fixes \( L \) (up to isomorphism), then \( \sigma_x \phi \sigma_x^{-1} \) fixes \( \sigma_x(L) \) for any exceptional point \( x \).}

Also the calculation of the Picard group reduces to the homogeneous case. Recall the following notion we already used before. Let \( H \) be an abelian group, \( h \in H \) and \( p \geq 2 \) be an integer. Then denote by \( H[p] \) the abelian group given by \( (H \oplus \mathbb{Z})/\mathbb{Z}(-h, p) \). Similarly, \( H[p_1, \ldots, p_t] \) is defined inductively.

The next proposition follows immediately with 6.1.5.

**Proposition 6.3.3.** With the same notations as in Proposition 6.3.1, we have

\[
\text{Pic}(\mathcal{X}) = \text{Pic}(\mathcal{X})\left[\frac{\sigma_{x_1}}{p_1}, \ldots, \frac{\sigma_{x_t}}{p_t}\right].
\]

The following is the extension of Proposition 5.2.3 to the weighted case.

**Proposition 6.3.4.** Let \( \mathcal{X} \) be an exceptional curve, such that for the underlying homogeneous situation there is an exhaustive automorphism in the Picard group, and such that the underlying bimodule is not of orbit case IIIb. Let \( \mathcal{G} \) be the ghost group. Then the group \( \text{Pic}(\mathcal{X})/\text{Pic}(\mathcal{X}) \cap \mathcal{G} \) acts simply transitive on the \( \text{Aut}(\mathcal{H}) \)-orbit of the structure sheaf \( L \), and there is a split exact sequence of groups

\[
1 \to \text{Pic}(\mathcal{X})/\text{Pic}(\mathcal{X}) \cap \mathcal{G} \to \text{Aut}(\mathcal{H})/\mathcal{G} \to \text{Aut}(\mathcal{X})/\mathcal{G} \to 1.
\]
Proof. Let $L'$ be lying in the same $\text{Aut}(\mathcal{H})$-orbit $X$ as $L$. After applying suitable shifts associated to exceptional points we can assume that $L$ and $L'$ are special with respect to the same set of exceptional simple objects. By perpendicular calculus, $L$ and $L'$ are line bundles over the associated homogeneous curve. By assumption, there is a Picard element mapping $L$ onto $L'$. Hence $\text{Pic}(\mathfrak{X})$ acts transitively on $X$. Each ghost fixes $L$, hence also any other member of $X$. We get an induced action of $\text{Pic}(\mathfrak{X})/\text{Pic}(\mathfrak{X}) \cap \mathcal{G}$ on $X$, which is obviously simply transitive.

Define $\text{Aut}(\mathcal{H})/\mathcal{G} \longrightarrow \text{Aut}(\mathfrak{X})/\mathcal{G}$ by $[\phi] \mapsto [\sigma \circ \phi]$ where $\sigma \in \text{Pic}(\mathfrak{X})$ such that $\sigma(\phi(L)) \simeq L$. This induces the split exact sequence. \hfill $\square$

In the special situation $k = \mathbb{R}$ Proposition 6.3.4 can be formulated as follows.

**Proposition 6.3.5.** Let $\mathfrak{X}$ be an exceptional curve over the real numbers. Let $\mathcal{G}$ be the ghost group. Then the group $\text{Pic}(\mathfrak{X})/\mathcal{G}$ acts simply transitive on the set of all (isomorphism classes of) special line bundles and there is a split exact sequence of groups

$$1 \longrightarrow \text{Pic}(\mathfrak{X})/\mathcal{G} \longrightarrow \text{Aut}(\mathcal{H})/\mathcal{G} \longrightarrow \text{Aut}(\mathfrak{X})/\mathcal{G} \longrightarrow 1.$$ \hfill $\square$

Note that [57, Lem. 4+Thm. 5] is not quite correct in the twisted case $\mathbb{C} \oplus \overline{\mathbb{C}}$, where $\mathcal{G}$ is non-trivial; moreover, we have to restrict to special line bundles as in the preceding proposition. We will give an example, where there are line bundles which are not special in 8.5.1.

For the domestic curves and the tubular curves over the real numbers the automorphism groups are listed in Appendix A.1.
CHAPTER 7

Exceptional objects

In this chapter we briefly expose two examples of problems in the context of exceptional objects. The first is the proof of the transitivity of the braid group action on the set of complete exceptional sequences over an exceptional curve which shows that the result is independent of the base field’s arithmetic. By contrast, the second example does not carry over to an arbitrary field. It deals with the characterization of exceptional curves by graded factoriality.

7.1. Transitivity of the braid group action

In this section we report on a joint result with H. Meltzer [59] which supports the philosophy that results on exceptional objects are essentially independent from the base field.

Let \( X \) be an exceptional curve with hereditary category \( \mathcal{H} \). A sequence \((E_1, \ldots, E_n)\) of exceptional objects in \( \mathcal{H} \) is called exceptional sequence, if for all \( j > i \) we have \( \text{Hom}(E_j, E_i) = 0 = \text{Ext}^1(E_j, E_i) \). It is called complete, if \( n \) coincides with the rank of the Grothendieck group of \( \mathcal{H} \).

The notion of complete exceptional sequences is closely related to the concept of tilting objects (complexes). We remark that there is a characterization of exceptional curves similar to 0.3.6 in terms of the existence of a complete exceptional sequence instead of a tilting object [67].

The braid group \( B_n \) on \( n \) strands is defined by generators \( \sigma_1, \ldots, \sigma_{n-1} \) and relations \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( j \geq i + 2 \) and \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) for \( i = 1, \ldots, n-2 \).

The braid group \( B_n \) acts on the set of all exceptional sequences of length \( n \): \( \sigma_i \) replaces in \((E_1, \ldots, E_n)\) the pair \((E_i, E_{i+1})\) by the pair \((E_{i+1}, R_{E_{i+1}}(E_i))\), where \( R_{E_{i+1}}(E_i) \) is the right mutation of \( E_i \) by \( E_{i+1} \) (see [59] for details).

**Theorem 7.1.1 ([59])**. Let \( X \) be an exceptional curve and let \( n \) be the rank of the Grothendieck group of \( X \). Then the braid group \( B_n \) acts transitively on the set of complete exceptional sequences in \( \mathcal{H} \).

Exceptional vector bundles play an important role in algebraic geometry in the study of vector bundles over various projective varieties, and
were introduced by Drezet and Portier [30] in connection with the investigation of stable bundles. Exceptional sequences and the braid group action were studied by Bondal [9]. Transitivity of this action on the set of complete exceptional sequences was shown for \( \mathbb{P}^2 \) by Rudakov [96], for \( \mathbb{P}^1 \times \mathbb{P}^1 \) by Rudakov [95] and for arbitrary del Pezzo surfaces by Kuleshov and Orlov [51]. We remark that there is a related concept of (collections of) spherical objects in the context of mirror symmetry, see [99].

In representation theory the importance of exceptional objects is without question. The transitivity of the braid group action on the set of complete exceptional sequences was established over an algebraically closed field for the category of modules over a hereditary algebra by Crawley-Boevey [19] and for the category of coherent sheaves over a weighted projective line by Meltzer [77]. Ringel [92] simplified and extended Crawley-Boevey’s result to arbitrary hereditary Artin algebras. This last result gave a hint that Meltzer’s result should also be true for arbitrary exceptional curves. On the other hand, the proof for an algebraically closed field presented in [77] did not work over an arbitrary field. Moreover, the results in [52, 55] on tubular curves have shown that it is often not predictable whether new effects occur or not.

We now briefly sketch the idea of the proof from [59]. The proof is by induction on the rank \( n \) of the Grothendieck group \( K_0(\mathcal{X}) \). The following lemma is crucial.

**Lemma 7.1.2.** Let \( \mathcal{X} \) be a non-homogeneous exceptional curve. Then each complete exceptional sequence in \( \mathcal{H} \) can be shifted by the braid group to an exceptional sequence which contains a simple object.

**7.1.3 (Proof of Theorem 7.1.1).** Relying on the lemma, the proof of Theorem 7.1.1 is straight-forward by induction, like in the algebraically closed case: For \( n = 2 \), that is, if \( \mathcal{X} \) is homogeneous, the proof is easy. Assume \( n > 2 \). Then there is a “canonical” complete exceptional sequence \( \mathcal{C} = (C_1, \ldots, C_n) \) given by a certain tilting bundle in \( \mathcal{H} \) ([69]). Given any complete exceptional sequence \( \mathcal{E} = (E_1, \ldots, E_n) \) one can assume by the lemma that \( E_n = S \) is an exceptional simple object. By the special structure of \( \mathcal{C} \) we can also assume \( C_n = S \). Then we consider the right perpendicular category \( S^\perp \) of \( S \), which is an exceptional curve where the rank of the Grothendieck group is \( n - 1 \), and use the induction hypothesis.

The proof of Lemma 7.1.2 uses the following rank formula [59] which follows from [42]. Forming the right perpendicular category \( E^\perp \) to an exceptional vector bundle \( E \), we switch to a module category \( E^\perp \simeq \text{mod}(\Lambda) \), where \( \Lambda \) is hereditary (not necessarily connected) with \( n - 1 \) simple modules.
Proposition 7.1.4. Let $E$ be an exceptional object in $\mathcal{H}_\Lambda$. Denote by $S_1, \ldots, S_{n-1}$ a complete system of simple modules in $E^\perp$ and by $P_1, \ldots, P_{n-1}$ their projective covers. Then

$$\frac{\text{rk}(E)^2}{[\text{End}(E) : k]} = \sum_{i=1}^{n-1} \frac{\text{rk}(P_i) \cdot \text{rk}(S_i)}{[\text{End}(P_i) : k]}.$$

7.1.5 (Proof of Lemma 7.1.2). It is sufficient to show that in each orbit $O$ there is a complete exceptional sequence $E = (E_1, \ldots, E_n)$ such that $E_i$ is of finite length for some $i$. Assume that for the orbit $O$ this is not the case. Then in $O$ appears an exceptional vector bundle $E$, such that $\frac{\text{rk}(E)^2}{[\text{End}(E) : k]}$ is minimal. We can assume, that $E$ appears in $E$ with $E = E_n$. Forming $E^\perp$, the exceptional sequence $(E_1, \ldots, E_{n-1})$ is complete in $E^\perp$. Using the transitivity of the braid group action for mod($\Lambda$) proved by Ringel [92], the sequence $E$ can be shifted to the exceptional sequences $(P_1, \ldots, P_{n-1}, E)$ and $(S_{n-1}, \ldots, S_1, E)$, where the $P_i$ and the $S_i$ are as in the preceding proposition, suitably ordered with $P_1 = S_1$. From the rank formula we get the contradiction $\frac{\text{rk}(P_1)^2}{[\text{End}(P_1) : k]} < \frac{\text{rk}(E)^2}{[\text{End}(E) : k]}$. □

Over an arbitrary field $k$ the endomorphism ring of an exceptional object is a finite dimensional skew field over $k$ and need not to coincide with $k$ itself. Therefore the following corollary is an important consequence of the transitivity of the braid group action.

Corollary 7.1.6. The list of endomorphism skew fields appearing in a complete exceptional sequence in $\mathcal{H}$ is invariant. □

7.2. Exceptional objects and graded factoriality

If $k$ is algebraically closed then there is a relationship between the concept of graded factoriality and the existence of exceptional objects, as illustrated by the following results:

7.2.1. For a smooth projective curve $C$ over an algebraically closed field $k$, the following are equivalent [68]:

1. $C$ is of genus zero.
2. coh($C$) admits an exceptional object.
3. coh($C$) admits a tilting object.
4. There is a (commutative) $\mathbb{Z}$-graded factorial $k$-algebra $R$, affine of Krull dimension two, such that coh($C$) $\simeq$ mod$_0^\mathbb{Z}(R)$.

Moreover, it follows from [52] that this is also true for $k = \mathbb{R}$. 
7.2.2. A similar result which follows from [54, 66], see [67] is: Let $\mathcal{H}$ be a small abelian connected category over an algebraically closed field $k$. Then the following assertions are equivalent:

1. $\mathcal{H}$ is equivalent to the category of coherent sheaves over an exceptional curve.
2. $\mathcal{H}$ is of the form $\frac{\text{mod}^H(R)}{\text{mod}^H_0(R)}$ for a (commutative) $H$-graded factorial affine $k$-algebra $R$ of Krull dimension two, where $H$ is a finitely generated abelian group of rank one.

The results of Chapters 1 and 6 indicate that the implication (1)$\Rightarrow$(2) (replacing “commutative” by “noncommutative”) remains valid for an arbitrary base field (up to the insertion of weights into non-central prime elements). But the converse and also 7.2.1 is wrong in general, even in a commutative situation, as Lenzing pointed out in [68]:

Example 7.2.3. Let $k = \mathbb{F}_2$ and $R$ be the commutative $\mathbb{Z}$-graded algebra
$$\mathbb{F}_2[X, Y, Z]/(X^6 + Y^3 + Z^2 + X^2Y^2 + X^3Z),$$
where $\text{deg}(X) = 1$, $\text{deg}(Y) = 2$ and $\text{deg}(Z) = 3$. (Note that $R$ is not generated in degree zero and one.) Then $R$ is $\mathbb{Z}$-graded factorial and the quotient category $\frac{\text{mod}^\mathbb{Z}(R)}{\text{mod}^\mathbb{Z}_0(R)}$ is equivalent to the category $\text{coh}(C)$ of coherent sheaves of a smooth projective curve $C$ of genus one (and not zero). □

It would be interesting to characterize the class of (noncommutative) graded factorial algebras which are related to the exceptional curves.
CHAPTER 8

Tubular exceptional curves

Let $\Sigma$ be a concealed canonical algebra over a field $k$ with corresponding exceptional curve $X$. $\Sigma$ (and $X$) are called tubular if the radical of the Grothendieck group $K_0(\Sigma)$ is finitely generated abelian of rank two. Equivalently, the Coxeter transformation is of finite order. Note that the radical for non-tubular $\Sigma$ ($X$, respectively) has always rank one.

An exceptional curve $X$ is tubular if and only if its virtual genus \[ g_X = 1 + \frac{\varepsilon p}{2} \left( \sum_{i=1}^{t} d_i \left( 1 - \frac{1}{p_i} \right) - \frac{2}{\varepsilon} \right) \]
is one. (Here, $p$ is the least common multiple of the weights $p_1, \ldots, p_t$, compare also 0.4.5.) From this property, it is not surprising that tubular exceptional curves have a strong affinity to elliptic curves $T$. In both cases, $\text{coh}(X)$ and $\text{coh}(T)$, all indecomposable objects lie in tubes (in the language of representation theory).

More precisely, Atiyah’s classification [3] of vector bundles over an elliptic curve $T$ over an algebraically closed field $k$ shows that $\text{coh}(T)$ consists of a rational family of tubular families, each parametrized by $T$ and consisting of homogeneous tubes. (Note, that here “rational family” means “indexed by the rational numbers”.)

In [89] Ringel introduced the tubular (canonical) algebras over an algebraically closed field and showed that the indecomposable modules over such an algebra can be classified basically by a rational family of tubular families, each parametrized by the projective line $\mathbb{P}^1(k)$; in each tubular family there are finitely many non-homogeneous tubes. Accordingly, the (bounded) derived category (see [36]) of a tubular algebra consists entirely of tubes [41].

The connection between these geometric and representation theoretic results was given by Geigle and Lenzing when they introduced the weighted projective lines [33] and later by Lenzing and Meltzer for the tubular case [70]. The fundamental concept there was that of tubular mutations [78], which are automorphisms of the derived category. (In this tubular situation the tubular shifts form a very small subclass of them.)

An additional feature of the mentioned results over an algebraically closed field is that for a fixed tubular algebra (tubular exceptional curve, elliptic
curve, respectively) all tubular families are equivalent categories. We showed in [52, 55] that this is no longer true over an arbitrary field. There are tubular exceptional curves which admit tubular families which are not equivalent. Accordingly, there are tubular exceptional curves $X$ and $X'$ which are derived equivalent but not isomorphic. In different terminology, $X$ and $X'$ are Fourier-Mukai partners.

The present chapter is devoted to the study of the automorphism group $\text{Aut}(D^b(X))$ of the bounded derived category of a tubular exceptional curve $X$. This group acts on the set of all separating tubular families in $D^b(X)$. Over an algebraically closed field the preceding remark implies that this group action is transitive, but over an arbitrary field there may occur more than one orbit. The number of these orbits is called the index of $X$. Our main result in [58] is that the index of a tubular exceptional curve $X$ is at most three and that such curves of index three exist. We summarize a proof for this result and present the new Proposition 8.1.6 which improves the argument.

Following that, we study examples exploiting the results from the previous chapters. The examples show how the automorphism group $\text{Aut}(D^b(X))$ can be determined in principle.

The central example will be a tubular exceptional curve of index three. In this example, the Grothendieck group is of rank three. (In general, the rank of the Grothendieck group of a tubular exceptional curve is at least three and at most ten [65].) Tubular exceptional curves with this property are of particular interest. First of all, since there is only one exceptional tube (of rank two) in each tubular family, exceptional objects are essentially determined by their slope and explicit calculations are much easier than for other tubular curves. This was demonstrated by Ringel [93], pointing out an interesting link between tilting modules and Farey fractions.

Moreover, in the tubular case the following effects arise only when $K_0(X)$ is of rank three:

- the occurrence of index three;
- the occurrence of roots (even 1-roots) in $K_0(X)$ which are not realizable by indecomposable objects in $\mathcal{H}$ (we refer to [52, 58]).

In the algebraically closed case each line bundle $L$ over an exceptional curve $X$ is exceptional. Over an arbitrary field this is also true for line bundles over a domestic exceptional curve (that is, when the virtual genus satisfies $g_X < 1$). We will show that it is false for some tubular cases where the Grothendieck group is of rank three or four.
8.1. Slope categories and the rational helix

Throughout this section let \( X \) be a tubular exceptional curve over a field \( k \).

8.1.1 (Slope). For each \( x \in K_0(X) \) such that \( \text{rk} x \neq 0 \) or \( \deg x \neq 0 \) define the slope by \( \mu x = \frac{\deg x}{\text{rk} x} \). The slope of a non-zero object in \( \mathcal{H} \) is defined as the slope of its class. \( \text{Stability} \) and \( \text{semistability} \) of non-zero objects in \( \mathcal{H} \) is defined with respect to the slope as in [33]. For each \( q \in \hat{\mathbb{Q}} = \mathbb{Q} \cup \{ \infty \} \) denote by \( \mathcal{H}^{(q)} \) the full subcategory of \( \mathcal{H} \) which is formed by the zero sheaf and the semistable sheaves of slope \( q \). We call the categories \( \mathcal{H}^{(q)} \) (and also their translates in the derived category) \( \text{slope categories} \). Note that for example \( \mathcal{H}^{(\infty)} = \mathcal{H}_0 \).

8.1.2. Since each indecomposable object in \( \mathcal{H} \) is semistable (compare [33, 5.5]), \( \mathcal{H} \) is the additive closure of its slope categories, and since \( \mathcal{H} \) is hereditary we have

\[
\mathcal{D} := D^b(X) = \bigvee_{n \in \mathbb{Z}} \mathcal{H}[n] = \bigvee_{(n,q) \in \mathbb{Z} \times \hat{\mathbb{Q}}} \mathcal{H}^{(q)}[n],
\]

which means that \( D^b(X) \) is the additive closure of the (disjoint) copies \( \mathcal{H}[n] \) and also of the \( \mathcal{H}^{(q)}[n] \), and moreover, there are non-zero morphisms from \( \mathcal{H}[n] \) to \( \mathcal{H}[n'] \) (from \( \mathcal{H}^{(q)}[n] \) to \( \mathcal{H}^{(q')}[n'] \), resp.) only if \( n \leq n' \) \((n, q) \leq (n', q')\), resp., where the \( \text{rational helix} \ \mathbb{Z} \times \hat{\mathbb{Q}} \) is endowed with the lexicographical order \([71]\)). More precisely, for all \( X, Y \in \mathcal{H} \) and all \( m, n \in \mathbb{Z} \) we have \( \text{Hom}_\mathcal{D}(X[n], Y[m]) = \text{Ext}^{m-n}_\mathcal{H}(X, Y) \). Note that \( \text{Ext}^i_{\mathcal{H}}(-, -) = 0 \) for \( i \in \mathbb{Z}, i \neq 0, 1 \). Here, \( X[n] \) denotes the element in the copy \( \mathcal{H}[n] \) which corresponds to \( X \in \mathcal{H} \). The automorphism \( T \) on \( \mathcal{D} \), which is induced by the assignment \( X \mapsto X[1] \), is called \text{translation functor}.

8.1.3 (Riemann-Roch formula). Let \( p \) be the least common multiple of the weights \( p_1, \ldots, p_t \). Recall that for any \( x, y \in K_0(X) \)

\[
\langle \langle x, y \rangle \rangle = \sum_{j=0}^{p-1} \langle \tau^j x, y \rangle.
\]

For any \( x, y \in K_0(X) \) the following formula holds ([65, 69]).

\[
\langle \langle x, y \rangle \rangle = \kappa \epsilon \left| \begin{array}{cc}
\text{rk} x & \text{rk} y \\
\deg x & \deg y
\end{array} \right|,
\]

which in case \( \text{rk} x \neq 0 \neq \text{rk} y \) can also be written as \( \langle \langle x, y \rangle \rangle = \kappa \epsilon \text{rk} x \text{rk} y (\mu y - \mu x) \). As application one gets: If \( X, Y \in \mathcal{H} \) are indecomposable with \( \mu(X) < \mu(Y) \), then \( \text{Hom}(X, \tau^j Y) \neq 0 \) for some \( j \).
8. TUBULAR EXCEPTIONAL CURVES

PROBLEM 8.1.4 (Calabi-Yau property). If $k$ is algebraically closed then
the Auslander-Reiten translation, that is, the Serre functor $\tau$ on $\mathcal{H}$ is of
order $p$ in the group Aut($\mathcal{H}$) (where $p$ is the least common multiple of the
weights). By Serre duality we conclude that the triangulated category $D^b(\mathcal{X})$
is Calabi-Yau of fractional dimension $1/p$, in the sense of [47].

It is an interesting question whether this is also always true if $k$ is an
arbitrary field. A priori on has to take ghosts into account.

8.1.5 (Interval categories). For each $q \in \overline{\mathbb{Q}}$ let $\mathcal{H}(q)$ be the subcategory
in $\mathcal{D}$ defined by

$$\mathcal{H}(q) = \mathcal{H}(q)[1] \vee \mathcal{H}_+^{(q)} \vee \mathcal{H}^{(q)},$$

where

$$\mathcal{H}_+^{(q)} = \bigvee_{-\infty < r < q} \mathcal{H}^{(r)}, \quad \mathcal{H}^{(q)} = \bigvee_{q < r \leq \infty} \mathcal{H}^{(r)}.$$  

Moreover

$$\mathcal{D} = \mathcal{D}^{(q)}_+ \vee \mathcal{H}^{(q)} \vee \mathcal{D}^{(q)}_-, $$

where $\mathcal{D}^{(q)}_+ = \{ X \in \mathcal{D} \mid \text{Hom}(\mathcal{H}^{(q)}, X) = 0 \}$ and $\mathcal{D}^{(q)}_- = \{ Y \in \mathcal{D} \mid \text{Hom}(Y, \mathcal{H}^{(q)}) = 0 \}$. We call the categories $\mathcal{H}(q)$ and also their translates
in $\mathcal{D}$ interval categories.

The first proof of the following proposition was a by-product in [55, 58]
of the case by case analysis how the automorphism group Aut($\mathcal{D}$) acts on
the set of slope categories. We now give a more systematic argument.

PROPOSITION 8.1.6. Let $\mathcal{X}$ be a tubular exceptional curve. For each $q \in \overline{\mathbb{Q}}$
the interval category $\mathcal{H}(q)$ is the sheaf category of a tubular exceptional curve
$\mathcal{X}(q)$.

PROOF. The key-point (see [55, Prop. 7]) is to show that the slope cate-
gory $\mathcal{H}^{(q)}$ is non-trivial. We modify the argument given in [55]. There is a
normalized rank function $\text{rk}_q$ on $K_0(\mathcal{X})$ such that $\text{rk}_q(F) \geq 0$ for all objects
$F \in \mathcal{H}(q)$ and $\text{rk}_q(F) = 0$ if and only if $F \in \mathcal{H}^{(q)}$. In fact, if $q = d/r$, where $d$
and $r$ are coprime, then (up to normalization) $\text{rk}_q(F) = d \cdot \text{rk}(F) - r \cdot \deg(F)$. Moreover, by semistability there is no non-zero morphism from an object of
rank zero to an object of non-zero rank. Since $\mathcal{H}^{(q)}$ is noetherian (possibly
trivial), noetherianness of the category $\mathcal{H}(q)$ follows straight-forwardly,
see [73, Lem. 5.2].

Let $\mathcal{H}(q)_0$ be the subcategory in $\mathcal{H}(q)$ of objects of finite length which is
a Serre subcategory. By [73] the quotient category $\mathcal{H}(q)/\mathcal{H}(q)_0$ is a length
category, and its length function defines a rank function on $\mathcal{H}(q)$. This
rank function is (up to some normalization factor) of the form $\text{rk}_{q'}$ for some
slope $q'$ (see [56, Prop. 5.3]). It follows, that $q' \leq q$ and (up to translation)
8.1. SLOPE CATEGORIES AND THE RATIONAL HELIX

\( H(q)_0 = H(q') \), which is non-trivial by noetherianness of \( H(q) \). Assume that \( q' < q \). Then:

(i) By semistability and noetherianness there is no non-zero object of slope \( r \) with \( q' < r \leq q \). (If \( 0 \neq F \) has slope \( r \) consider a maximal subobject of \( F \); the simple factor then lies in \( H(q') \) which gives a non-zero map from \( F \) to an object of smaller slope.)

(ii) Since the category \( H(q') \) is connected there is a non-zero torsion-free object in \( H(q') \). Shifting this object sufficiently far to the left leads to a non-zero object of slope \( r \) with \( q' < r \leq q \), a contradiction to (i).

Therefore \( q = q' \) follows, hence \( H(q) \) is also non-trivial. \( \square \)

It follows that the slope induces a bijection between all slope categories in \( D^b(X) \) and the elements of the rational helix \( \mathbb{Y} = \mathbb{Z} \times \hat{\mathbb{Q}} \).

8.1.7. Moreover, it follows that for all \( q \in \hat{\mathbb{Q}} \) there is defined the \( q \)-symbol, that is, the symbol of the curve \( X(q) \).

For all \( q, q' \in \hat{\mathbb{Q}} \) the tubular exceptional curves \( X(q) \) and \( X(q') \) are derived equivalent, that is, they are Fourier-Mukai partners. In particular, all the Grothendieck groups \( K_0(X(q)) \) and \( K_0(X(q')) \) (equipped with the Euler forms) are isomorphic. This means, by definition, that the symbols of \( X(q) \) and \( X(q') \) are equivalent, but in general they are different, and accordingly the curves \( X(q) \) and \( X(q') \) non-isomorphic, that is, \( H(q) \neq H(q') \). (Compare the list of (equivalence classes of) tubular symbols given in Appendix B.)

Furthermore, for all \( q \) the tubular shifts associated to points in \( X(q) \) are defined and are automorphisms of \( H(q) \), which extend to elements in \( \text{Aut}(D) \). These are by definition the tubular mutations.

**LEMMA 8.1.8 ([55, Cor. 11]).** Let \( \phi \in \text{Aut}(D) \). For any element \((n, q)\) of the rational helix there is a unique \((n', q')\) in the rational helix such that \( \phi(H(q)[n]) = H(q')[n'] \). Hence, by setting \( \widehat{\phi}(n, q) = (n', q') \) we get an automorphism \( \widehat{\phi} \) of the rational helix. This induces a homomorphism of groups \( \Phi : \text{Aut}(D) \rightarrow \text{Aut}(\mathbb{Y}) \), \( \phi \mapsto \widehat{\phi} \).

8.1.9. Note that \( \text{Aut}(\mathbb{Y}) \simeq B_3 \), the braid group on three strands, which is defined by generators \( s, \ell \) and the relation \( s\ell s = \ell s\ell \) [71]. The translation \( T \) is mapped under \( \Phi \) to \( t = (s\ell)^3 \), which is a central element of infinite order. We have the exact sequence

\[ 1 \longrightarrow \langle t \rangle \longrightarrow B_3 \overset{p}{\longrightarrow} \text{PSL}_2(\mathbb{Z}) \longrightarrow 1, \]

given by \( \ell \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \ s \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \).

We described in [58] which subgroups of \( B_3 \) occur as images of \( \Phi \) in the different cases (compare also [56, Table 1]). Typically the image of
Φ is generated by the images of two or three tubular mutations and the translation in the derived category. Here we write “typically”, since the situation is not fully clarified in the cases where the rank of the Grothendieck group is three. See also Remark 8.2.9 below.

8.1.10. The kernel of Φ is given by the automorphisms preserving the slope. Since these automorphisms restrict naturally to \( \mathcal{H} \), we consider them as elements in Aut(\( \mathcal{H} \)), which defines the subgroup Aut\(_{\mu}(\mathcal{H})\). With the assumptions of Proposition 6.3.4 the slope preserving automorphisms are just those of Pic\(_0(\mathcal{X})\) and of Aut(\( \mathcal{X} \)) (and compositions of them), and with the ghost group \( \mathcal{G} \) we have

\[
\text{Aut}_{\mu}(\mathcal{H})/\mathcal{G} \cong \left( \text{Pic}_0(\mathcal{X})/\text{Pic}_0(\mathcal{X}) \cap \mathcal{G} \right) \rtimes \text{Aut}(\mathcal{X})/\mathcal{G}.
\]

In particular it follows, that if \( \mathcal{X} \) and \( \mathcal{X}' \) are derived equivalent tubular exceptional curves, then there is a relationship between Pic\(_0(\mathcal{X})\) and Aut(\( \mathcal{X} \)) on the one hand and Pic\(_0(\mathcal{X}')\) and Aut(\( \mathcal{X}' \)) on the other hand. But it is not true in general that the automorphism groups of \( \mathcal{X} \) and \( \mathcal{X}' \) are isomorphic, neither the geometric automorphism groups.

8.2. The index of a tubular exceptional curve

We keep the notations from the previous section.

If \( k \) is algebraically closed the map Φ is always surjective. Equivalently, Aut(D) acts transitively on \( \mathcal{Y} \) in this case. But in general Φ is not surjective.

Definition 8.2.1. Let \( \mathcal{X} \) be a tubular exceptional curve. Then the number of Aut(D\(_b(\mathcal{X})\))-orbits in the rational helix \( \mathcal{Y} \) is called the index of \( \mathcal{X} \). Similarly, the index of a tubular algebra is defined.

Theorem 8.2.2 ([52, 58]). The index of a tubular exceptional curve is at most three.

In the following we sketch the idea of the proof. For details we refer to [58].

8.2.3. Denote by \( V = K_0(\mathcal{X}) \) the Grothendieck group of \( \mathcal{H} \), by \( R = \text{Rad}(V) \) the radical of \( V \), and by \( \mathbb{P}R \) the set of direct summands of \( R \) of rank one. Taking the slope of generators of such direct summands induces a bijection between \( \mathbb{P}R \) and \( \hat{\mathbb{Q}} \) (see [56]). There is the following commutative diagram

\[
\begin{array}{ccc}
\text{Aut}(D) & \xrightarrow{\Phi} & \text{Aut}(\mathcal{Y}) = B_3 \\
\downarrow{\kappa} & & \downarrow{\nu} \\
\text{Aut}(V) & \xrightarrow{\Psi} & \text{Aut}(\mathbb{P}R) = \text{PSL}_2(\mathbb{Z}).
\end{array}
\]
Each element in $\text{Aut}(V)$ induces by restriction to the radical an automorphism of $\mathbb{P}R$, which defines the map $\Psi$. The automorphism group of $\mathbb{P}R$ can be identified with the projective modular group $\text{PSL}_2(\mathbb{Z})$ (see [56]). $\kappa$ is defined by $\kappa(\phi)([X]) = [\phi X]$ for any $\phi \in \text{Aut}(\mathcal{D})$ and any $X \in \mathcal{D}$.

8.2.4. It is shown in [56] that the group $\text{Aut}(V)$ acts on $\hat{\mathbb{Q}}$ with at most two orbits. This is shown by determining a subgroup of $\text{Aut}(V)$ generated by a few certain shift automorphisms (defined on the K-theoretical level, see [65]) such that the $\Psi$-image of this subgroup coincides with the $\Psi$-image of $\text{Aut}(V)$ and therefore this subgroup acts with at most two orbits. Then the idea of the proof of Theorem 8.2.2 is to realize these K-theoretical automorphisms by tubular mutations on the derived level. In the cases where the rank of the Grothendieck group is greater or equal than four this can be accomplished without problem, so that there are also at most two orbits on the derived level. But if the rank of the Grothendieck group equals three (that is, in each tubular family there is precisely one exceptional tube, and this is of rank two) the index depends also on the arithmetic of the base field, which leads to cases of index three (see Proposition 8.2.5 below; see also Appendix B for a list of the 17 tubular cases). Moreover, the analysis in [58] shows the following propositions which stress the special role of the rank three case.

**Proposition 8.2.5.** Let $X$ be a tubular exceptional curve such that the Grothendieck group is of rank three. Then the index of $X$ is at most three.

Assume that there exists $q \in \hat{\mathbb{Q}}$ such that the numerical type of $X(q)$ is $\varepsilon = 1$ and such that there exists a unirational point in $X(q)$. Then the index of $X$ is at most two.

If the symbol of $X$ equals $\left( \begin{array}{c} 2 \\ 4 \\ 2 \end{array} \right)$ and if, for example, $X$ and $X(2)$ contain unirational points, then the index is one.

**Proof.** Let $q \in \hat{\mathbb{Q}}$ such that the numerical type of $X(q)$ is $\varepsilon = 1$. Switching to $X(q)$ we can assume that the numerical type of $X$ itself is $\varepsilon = 1$. The $\Phi$-image of the tubular mutations with respect to the exceptional tubes containing the structure sheaf $L$ and an exceptional simple object together with the $\Phi$-image of the translation $T$ gives the subgroup $\langle t, \ell, s^4 \rangle$ of $B_3$. This subgroup acts with three orbits on the rational helix. The remaining assertions follow from the analysis in [56, 10.].

It is easy to see that each $(2, 2)$-bimodule over a finite field $k$ is non-simple, hence there is a unirational point for the associated curve (compare 0.6.2). The same is true for the field $k = \mathbb{R}$ of real numbers. Thus we get
Corollary 8.2.6. The index of a tubular algebra (or tubular exceptional curve) over any finite field $k$, or over the field $\mathbb{R}$ of real numbers, is at most two.

Proposition 8.2.7 ([58]). Let $\mathbb{X}$ be a tubular exceptional curve and $r$ be the rank of its Grothendieck group. There is a subgroup $U$ of $\text{Aut}(\text{D}^b(\mathbb{X}))$ acting transitively on each $\text{Aut}(\text{D}^b(\mathbb{X}))$-orbit in the set of all slope categories such that additionally the following holds:

If $r \geq 4$, or if the index is three, then $U$ is generated by the translation $T$ and two tubular mutations associated to exceptional tubes.

Proposition 8.2.8. Let $\mathbb{X}$ be a tubular exceptional curve such that the Grothendieck group is of rank three.

1. Let $U$ be the subgroup of $\text{Aut}(\text{D}^b(\mathbb{X}))$ which is generated by the translation $T$ and all tubular mutations associated to exceptional tubes (for all slopes). Then $U$ acts with three orbits on the set of all slope categories.

2. Moreover, for $K_0(\mathbb{X})$ there are three possible cases:
   a) The symbol is \[
   \begin{pmatrix}
   2 \\
   4 \\
   2
   \end{pmatrix}. 
   \] Then the index of $\mathbb{X}$ is one, two or three (the precise value depending on the arithmetic of $k$).
   b) The symbol is \[
   \begin{pmatrix}
   2 \\
   4 \\
   4
   \end{pmatrix} \text{ or } \begin{pmatrix}
   2 \\
   2 \\
   2
   \end{pmatrix}. 
   \] Then the index of $\mathbb{X}$ is two or three.
   c) The symbol is \[
   \begin{pmatrix}
   2 \\
   4 \\
   2
   \end{pmatrix} \text{ or } \begin{pmatrix}
   2 \\
   2 \\
   2
   \end{pmatrix}. 
   \] Then the index of $\mathbb{X}$ is two or three.

Remark 8.2.9. (1) For the case c) in the preceding proposition we constructed examples for indices two and three, respectively ([58]; see the following sections). We remark that in [58] we stated that in cases a) and b) examples of index three do not exist. This was based on an argument which turned out to be wrong. We now do not see any reason why examples of index three in these cases should not exist.

2. Which index actually occurs depends on the question which of the 6 cosets of the subgroup $\langle t, \ell, s^4 \rangle$ in $B_3$ have representatives which are realizable by automorphisms of the derived category. If (besides the identity) none of these representatives is realizable the index is three. If additionally only $s^2$ is realizable then the index is two. If any other coset is realizable then the index is one (only possible in case a). Another question (related to Problem 5.2.2) is whether such realizations (assuming existence) are always possible by tubular mutations (compare Proposition 6.3.4).
8.3. A tubular curve of index three

In this section we exhibit our example [58] of a tubular exceptional curve \( X \) of index three and list its further properties. Knowledge of the action of \( \text{Aut}(D^b(X)) \) on the rational helix allows to determine \( \text{Aut}(D^b(X)) \) itself.

Recall that for sequences \( \mathbf{p} = (p_1, \ldots, p_t) \) and \( \mathbf{d} = (d_1, \ldots, d_t) \) of positive integers the abelian group \( \mathbb{Z}[\frac{d_1}{p_1}, \ldots, \frac{d_t}{p_t}] \) is also denote by \( \mathcal{L}(\mathbf{p}, \mathbf{d}) \).

**Proposition 8.3.1.** There is a tubular exceptional curve \( X \) over \( k = \mathbb{Q} \) such that the following holds:

1. The index of \( X \) is three.
2. A projective coordinate algebra of \( X \) is given by the graded factorial algebra
   \[
   R = \mathbb{Q}[X, Y, Z, U]/(X^2 + Y^2 + Z^2, U^2 - X^2 - 3Y^2),
   \]
   which is graded by \( \mathcal{L}(\mathbf{p}, \mathbf{d}) \) with \( \mathbf{p} = (1, 1, 1, 2) \) and \( \mathbf{d} = (1, 1, 1, 2) \). (This property uniquely determines \( X \).)
3. There is a tilting bundle whose endomorphism ring is the canonical algebra \( \Lambda \) given by the species
   \[
   \text{K}_M \text{Q} \text{K}_F
   \]
   plus certain relations, where \( K = \mathbb{Q}(\sqrt{-3}, \sqrt{2}) \), \( F = (\frac{-1,-1}{\mathbb{Q}}) \) be the skew field of quaternions over \( \mathbb{Q} \) on generators \( i \) and \( j \) with relations
   \[
   i^2 = -1, \quad j^2 = -1, \quad ij = -ji,
   \]
   and moreover, \( M \) is the bimodule from 5.7.3, that is, \( M = K(K \oplus K)_F \) with the canonical \( K \)-action and the \( F \)-action defined by
   \[
   (x, y) \cdot i = \frac{1}{\sqrt{-3}}(\sqrt{2}x + y, x - \sqrt{2}y), \quad (x, y) \cdot j = (y, -x)
   \]
   for all \( x, y \in K \).
4. For the automorphism groups we have \( \text{Aut}(X) \cong \mathbb{V}_4 \), the Klein four group, and
   \[
   \text{Aut}(D^b(X)) \cong (\mathbb{Z}_2 \times \mathbb{V}_4) \rtimes (F_2 \times \mathbb{Z}),
   \]
   where \( F_2 \) is the free group in two generators.

**Proof.** (2) Consider the tame bimodule \( kF_F \). The associated homogeneous exceptional curve admits \( S = \mathbb{Q}[X, Y, Z]/(X^2 + Y^2 + Z^2) = \mathbb{Q}[x, y, z] \) by 5.5.1 as projective coordinate algebra. The element \( x^2 + 3y^2 \) is a prime element in \( S \) (see [52, 3.10.1]). Insertion of the weight \( p = 2 \) into this prime leads to the \( H \)-graded factorial algebra
   \[
   R = \mathbb{Q}[X, Y, Z, U]/(X^2 + Y^2 + Z^2, U^2 - X^2 - 3Y^2),
   \]
where $H = L(p, d)$ is the abelian group as above, that is, generated by the degrees $\deg X, \deg Y, \deg Z, \deg U$ with relations $\deg X = \deg Y = \deg Z, 2 \deg U = 2 \deg X$. The torsion subgroup of $H$ is generated by $\deg U - \deg X$.

Denote by $X$ the corresponding exceptional curve with hereditary category $H = \text{coh}(X) = \text{mod} H(R)$.

Obviously, $X$ has the symbol $\sigma[X] = \begin{pmatrix} 2 & 2 & 2 & 2 \end{pmatrix}$, and hence is tubular.

(3) The construction in [69, Prop. 5.4] leads to a tilting bundle $L \longrightarrow L(1)$ whose endomorphism ring is the canonical algebra $\Lambda$ as described in (3): The endomorphism ring of the simple object, which corresponds to the prime element $x^2 + 3y^2$, is isomorphic to $\mathbb{Q}[X, Y]/(X^2 + Y^2 + 1, X^2 + 3Y^2) \simeq \mathbb{Q}(\sqrt{-3}, \sqrt{2})$. Moreover, by considering dimensions (see [65, Prop. 10.1]), $M$ is a $(2, 2)$-bimodule, moreover a simple bimodule, since $K \not\cong F$. Considering the isomorphism of algebras,

$$K \otimes_{\mathbb{Q}} F \simeq \begin{pmatrix} -1, -1 \\ K \end{pmatrix} \simeq M_2(K),$$

using that $M$ is a simple $K \otimes_{\mathbb{Q}} F$-module, it follows, that $M$ is the bimodule from 5.7.3.

(1) Since $L, L(1)$ and $L$ are exceptional having pairwise non-isomorphic endomorphism skew fields $\mathbb{Q}, K$ and $F$, respectively, the three corresponding tubular families are pairwise non-equivalent, hence the index of $X$ is three. In fact, each tubular family contains precisely one exceptional tube (of rank two) and therefore contains precisely two exceptional objects, having the same endomorphism skew field.

(4) Since the ghost group $\mathcal{G}$ is trivial, from 8.1.10 we get the following exact sequence of groups

$$1 \longrightarrow \text{Pic}_0(X) \times \text{Aut}(X) \longrightarrow \text{Aut}(D^b(X)) \longrightarrow B_3,$$

where $\text{Pic}_0(X)$ is isomorphic to the torsion part of the grading group $H$, hence to $\mathbb{Z}_2$. Moreover, $\text{Aut}(X)$ is the automorphism group of $X$, which by 6.3.1 consists of the automorphisms of the projective spectrum $\mathbb{Y}$ of the graded factorial algebra $S$ fixing the point corresponding to the prime element $x^2 + 3y^2$. By 5.5.1, $\text{Aut}(\mathbb{Y}) \simeq \text{SO}_3(\mathbb{Q})$, and this group acts on prime elements of degree one (which are of the form $\alpha x + \beta y + \gamma z$) like a matrix on $(\alpha, \beta, \gamma)$. In particular, each automorphism of $\mathbb{Y}$ is uniquely determined by its action on the points of degree one. Each such automorphism thus extends uniquely to a graded algebra automorphism of $S$. Then it is easy to check, which
automorphisms are fixing the prime ideal generated by $x^2 + 3y^2$, and this yields $\text{Aut}(X) \simeq \mathbb{V}_4$, the Klein four group.

We determine the image of $\Phi$: Consider the tubular mutations $\sigma_L$ and $\sigma_S$ which are associated to the tube containing $L$ (of slope 0) and to the tube containing an exceptional simple object $S$ (of slope $\infty$), respectively. By [56, 6] these maps on slopes induce the actions $q \mapsto q/(1 - 2q)$ and $q \mapsto q + 2$, respectively. It follows that the image of $\Phi$ is given by $\langle t, s^2, \ell^2 \rangle$ (with $t = (s\ell)^3$), which is a subgroup in $B_3$ of index 6. It follows from [98] that $\langle s^2, \ell^2 \rangle$ is isomorphic to $F_2$ and $t$ is central, hence we get $\text{Im} \Phi \simeq F_2 \times \mathbb{Z}$.

It is easy to see, that the induced exact sequence of groups

$$1 \longrightarrow \text{Pic}_0(X) \rtimes \text{Aut}(X) \longrightarrow \text{Aut}(\text{D}^b(X)) \longrightarrow \text{Im} \Phi \longrightarrow 1,$$

splits. Then the result follows.

8.3.2. We keep the notation of the proposition. There are two companion curves which are derived equivalent to $X$, namely the curves $X(0)$ and $X(1)$ (in the slopes $q = 0$ and $q = 1$, respectively). Moreover, $X$, $X(0)$ and $X(1)$ are pairwise non-isomorphic.

This corresponds to the fact, that there are two (further) tilting bundles in $\mathcal{H}$, such that the endomorphism rings are the canonical algebras $\Lambda(0)$ and $\Lambda(1)$ given by the species

$$
\begin{array}{c}
\text{Q} \\
F \\
M^* \quad \text{Q} \\
\end{array}
\xymatrix{ & \text{Q} \\
F \ar[r] & M^* \ar[r] & K \ar[l]}
$$

(where $F^*$ and $M^*$ denote the dual bimodules) and

$$
\begin{array}{c}
\text{Q} \\
F \\
K \quad \text{Q} \\
\end{array}
\xymatrix{ & \text{Q} \\
F \ar[r] & N \ar[r] & K \ar[l]}
$$

(plus relations), respectively (for some bimodule $N$), see [58]. The algebras $\Lambda$, $\Lambda(0)$, and $\Lambda(1)$ are tilting equivalent.

It follows from 1.7.12 that for $X(1)$ one gets as projective coordinate algebra a graded algebra, arising by inserting the weight $p = 2$ into some prime element of

$$\mathbb{Q}(X, Y, Z)/(XY - YX, XZ - ZX, YZ + ZY, Z^2 - 3Y^2 - 2X^2).$$

8.3.3. In [72] the following is shown over an algebraically closed field: Two finite dimensional algebras $\Lambda$ and $\Lambda'$ which are derived equivalent to the same tubular exceptional curve and having the same Cartan matrix are isomorphic.
This is not true in general over arbitrary fields, since the tilting equivalent tubular canonical algebras $\Lambda$ and $\Lambda\langle 1 \rangle$ as above have the same Cartan matrix, but are obviously not isomorphic.

### 8.4. A related tubular curve of index two

The next example shows that the index is not a K-theoretic invariant.

**Proposition 8.4.1.** There is a tubular exceptional curve $X'$ over the field $k = \mathbb{Q}(i)$ such that the following holds:

1. With the tubular curve $X$ from 8.3.1, the Grothendieck groups $K_0(X')$ and $K_0(X)$, equipped with the Euler form, are isomorphic.
2. The index of $X'$ is two.
3. There is a graded factorial coordinate algebra of $X'$ which arises by insertion of the weight $p = 2$ into the central prime element $X^4 - Y^4$ in the twisted polynomial algebra $K[X; Y, \alpha]$, where $K = k(\sqrt{2})$ and $\alpha$ is the $k$-automorphism $\sqrt{2} \mapsto i\sqrt{2}$.
4. There is a tilting bundle whose endomorphism ring is the canonical algebra $\Lambda'$ given by the species

\[
\begin{array}{ccc}
K & \xrightarrow{M} & K \\
\downarrow & & \downarrow \\
K & \xrightarrow{M} & K
\end{array}
\]

plus certain relations, where $M$ is the non-simple bimodule $M(K, \alpha)$.

5. $\text{Aut}(X') \simeq \mathbb{Z}_4$ coincides with the ghost group (hence the geometric automorphism group is trivial), and for $\text{Aut}(\mathcal{D}^b(X'))$ there is the exact sequence of groups

\[1 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_4 \rightarrow \text{Aut}(\mathcal{D}^b(X')) \rightarrow U \rightarrow 1,
\]

where $U$ is the subgroup $(s^2, \ell)$ of the braid group $B_3$.

**Proof.**

3. The field extension $K/k$ is Galois with cyclic Galois group generated by $\alpha$. Let $X'_0$ be the homogeneous curve associated to the bimodule $M = M(K, \alpha)$. A projective coordinate algebra is given by $K[X; Y, \alpha]$. Insertion of the weight $p = 2$ into the central prime element $\pi = X^4 - Y^4$ leads by 6.2.4 to an exceptional curve $X'$. By 1.7.10 the multiplicity of the inserted point is 4, and hence the symbol of $X'$ is $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $X'$ is tubular.

1. This follows, since the symbols $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} 2$ are equivalent [56].

2. Since $M$ is non-simple, by 8.2.5 the index is two.

4. This follows, since by 1.7.10 the endomorphism ring of the exceptional simple object is $k$. 

(5) By 5.3.4, \( G \cong \mathbb{Z}_4 \), and for any transformation \( Y \mapsto aY \) on \( X'_0 \) with \( a \in \mathbb{Q}(i)^* \) leaving the prime ideal generated by \( X^4 - Y^4 \) fixed, \( N(a) = 1 \) follows and \( Y \mapsto aY \) is trivial. Thus \( \text{Aut}(X') = G \).

Consider the map \( \Phi : \text{Aut}(D_0(X')) \rightarrow B_3 \). Take the tubular mutations \( \sigma_L \) and \( \sigma_S \) which are associated to the tube containing \( L \) (of slope 0) and to a homogeneous tube (containing a simple object \( S \) of slope \( \infty \)), respectively. By [56, 6] these maps on slopes induce the actions \( q \mapsto q/(1 - q) \) and \( q \mapsto q + 2 \), respectively. It follows that the image of \( \Phi \) is given by \( \langle s^2, \ell \rangle \). (This subgroup in \( B_3 \) is of index three, and described by the defining relation \( (s^2 \ell)^2 = (\ell s^2)^2 \), compare [57].)

By 8.1.10 the kernel of \( \Phi \) is generated by \( G \) and the generator of \( \text{Pic}_0(X') \) of order two. By 0.4.8 this generator commutes with all ghosts, and thus the kernel of \( \Phi \) is given by \( \text{Pic}_0(X') \times G \). \( \square \)

8.5. Line bundles which are not exceptional

Example 8.5.1. Let \( k = \mathbb{R} \) and let \( X \) be the tubular exceptional curve arising by inserting the weight \( p = 2 \) into the central prime element \( \pi = X^4 + Y^4 \) in \( \mathbb{C}[X, Y] \). Then there is a line bundle \( L' \), which lies in a homogeneous tube.

Proof. Let \( S \) be one (of the two) exceptional simple object in \( \mathcal{H} \) such that \( \text{Hom}(L, S) \neq 0 \). We get a short exact sequence

\[
0 \rightarrow L' \rightarrow L \rightarrow S \rightarrow 0,
\]

where \( L' \) is a line bundle. By 5.6.1 the exceptional point \( x \) has symbol data \( f(x) = 2, e(x) = 2 \), and then \( \langle [L'], [L'] \rangle = 0 \). But then \( \text{Ext}^1(L', L') \neq 0 \). Since line bundles are stable, \( L' \) lies in the mouth of a homogeneous tube. \( \square \)

This fact is surprising, since in the commutative case graded factoriality implies that each line bundle is a degree shift of the structure sheaf \( L \) [33, Prop. 2.1]. We conclude, that this is not longer true in the noncommutative case, since the projective coordinate algebra \( R \), which arises by inserting the weight \( p = 2 \) into \( \pi = X^4 + Y^4 \) in \( \mathbb{C}[X, Y] \) is graded factorial.

8.5.2. We construct further examples of non-exceptional line bundles.

(1) In the same way as in 8.5.1 such a line bundle can be constructed for any tubular exceptional curve with symbol \( \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \) or \( \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} \), and similarly (taking \( S^2 \) as cokernel), for the symbol \( \begin{pmatrix} 2 \\ 4 \end{pmatrix} \).

(2) Consider a tubular curve \( X \) with symbol \( \begin{pmatrix} 3 \\ 3 \end{pmatrix} \). (Compare also Example 1.6.10.) For the kernel \( L' \) of an epimorphism \( L \rightarrow S \oplus S^{(2)} \) one gets
\( \langle [L'], [L'] \rangle = 0 \), where \( S \) is exceptional simple and \( S^{(2)} \) is the indecomposable middle term of the almost split sequence ending in \( S \). Such an epimorphism exists: The universal sequence over the underlying homogeneous curve

\[
0 \longrightarrow L(-x) \xrightarrow{x_L} L \longrightarrow S_x^3 \longrightarrow 0
\]

and the projections \( S_x^3 \to S_x^2 \to S_x \) lead to the epimorphism of 3-cycles

\[
j(L) = \left[ \begin{array}{cccc}
L & L & L & x_L \\
\downarrow & \downarrow & \downarrow & \downarrow \\
S \oplus S^{(2)} & S_x^2 & S_x & 0 \\
\end{array} \right]
\]

Representing \( L' \) as kernel, we see that \( L' \) stores an “irreducible” factorization of \( x_L \).

(3) For the symbol \( \left( \begin{array}{cc} 2 & 2 \\ 2 & 2 \end{array} \right) \) it is also possible to construct an example. Take again \( \mathbb{C}[X, Y] \) and take the central prime elements \( \pi_1 = Y^2 - X^2 = (Y - X)(Y + X) \) and \( \pi_2 = Y^2 - 4X^2 = (Y - 2X)(Y + 2X) \). The element \( (Y + X)(Y - 2X) \) induces a short exact sequence

\[
0 \longrightarrow L(-2) \longrightarrow L \longrightarrow S_1 \oplus S_2 \longrightarrow 0,
\]

where \( S_1 \) and \( S_2 \) are the simple objects concentrated in the points \( x_1 \) and \( x_2 \) associated to \( \pi_1 \) and \( \pi_2 \). This induces, after insertion of weights, a short exact sequence in \( H \left( \begin{array}{cc} 2 & 2 \\ x_1 & x_2 \end{array} \right) \)

\[
0 \longrightarrow L' \longrightarrow j(L) \longrightarrow S'_1 \oplus S'_2 \longrightarrow 0,
\]

where \( S'_1 \) and \( S'_2 \) are exceptional simple objects concentrated in different points and \( L' \) is a line bundle. It is easy to see, that \( \langle [L'], [L'] \rangle = 0 \).

**Proposition 8.5.3.** Let \( X \) be a tubular exceptional curve. If there is an exceptional point which is multiplicity free then each line bundle is exceptional. Accordingly, if there exists a line bundle \( L' \) which is not exceptional then the symbol of \( X \) is one of the five symbols in 8.5.2.

**Proof.** For an exceptional point \( x \) call a line bundle \( L' \) \( x \)-special, if \( L' \) maps onto precisely one simple object \( S_x \) in \( U_x \). If \( e(x) = 1 \), then each line bundle \( L' \) is \( x \)-special, which follows from the formula

\[
1 = \text{rk}(L') = \frac{1}{\kappa \varepsilon f(x)} \left( \langle [L'], [S_x] \rangle + \cdots + \langle [L'], [\pi^{p(x)-1} S_x] \rangle \right),
\]

and each summand inside the brackets is divisible by \( \langle [S_x], [S_x] \rangle = \kappa \varepsilon f(x) \) (compare 0.4.5).
If \( L' \) is \( x \)-special for some exceptional point \( x \), then \( L' \) is exceptional, since otherwise it would lie in a homogeneous tube and would be therefore \( \tau \)-stable. Therefore we can exclude all tubular curves where there is a multiplicity free, exceptional point, which one can detect from the symbol. Only for the five symbols from 8.5.2 such a point does not exist.

\[ \square \]

**Remark 8.5.4.** Let \( L' \) be a line bundle. If \( L \) is special with \( \deg(L) > \deg(L') \), then by the Riemann-Roch formula \( L' \) embeds into \( L \) (up to \( \tau \)-translations). If \( \deg(L) \) is minimal with this property then it is easy to check K-theoretically whether \([L']\) is a root or not. One gets that the examples above are essentially all examples of non-exceptional line bundles (in the tubular case).

**Remark 8.5.5.** Let \( X \) be the homogeneous exceptional curve with projective coordinate algebra \( R = \mathbb{C}[X,Y] \). Consider the non-exceptional line bundle \( L' \) as 2-cycle, concentrated in the point \( x \) corresponding to \( \pi = X^4 + Y^4 \), which decomposes as \( \pi = (X^2 - iY^2)(X^2 + iY^2) \) into irreducible elements \( u_1 \) and \( u_2 \):

\[
\begin{align*}
\mathfrak{T} &= [ \begin{array}{c} L \\ \downarrow \end{array} ] \\
L' &= [ \begin{array}{c} L \\ \downarrow \end{array} ] \\
\sigma \pi L &= [ \begin{array}{c} L \\ \downarrow \end{array} ]
\end{align*}
\]

\[ L \quad \xrightarrow{\pi} \quad L(4) \]

\[ L' \quad \xrightarrow{u_1} \quad L(4) \]

\[ \sigma \pi L \quad \xrightarrow{\pi} \quad L(4) \quad \xrightarrow{L(4)} \]

This construction can be done generally with an irreducible decomposition of a central prime (see 6.2.7). It would be interesting to understand, why this leads sometimes to exceptional line bundles \( L' \) and sometimes to non-exceptional \( L' \).
Automorphism groups over the real numbers

A.1. Tables for the domestic and tubular cases

If $k$ is algebraically closed and of tubular weight type $(2\ 2\ 2\ 2)$, then $X$ depends also on some parameter $\lambda \in k$, $\lambda \neq 0, 1$. More precisely, two such curves $X(2\ 2\ 2; \lambda)$ and $X(2\ 2\ 2; \mu)$ are isomorphic if and only if they have the same $j$-invariant $j(\lambda) = 2^6(\lambda^2 - \lambda + 1)^3/(\lambda^2(\lambda - 1)^2)$. Moreover, the automorphism group depends on this $j$-invariant 

$$\text{Aut}_X = \begin{cases} 
A_4 & j = 0, \\
D_4 & j = 1728, \\
V_4 & j \neq 0, 1728.
\end{cases}$$

Here, $A_4$ denotes the alternating group (which is of order 12), $D_4$ the dihedral group (of order 8) and $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ the Klein four group. In Table A.2 we denote by $\nabla \text{PGL}_2(\mathbb{R})$ the subgroup of $\text{PGL}_2(\mathbb{R})$ formed by the upper triangular matrices.

In the following tables (taken from [57]), we exhibit the automorphism groups for the domestic and for the tubular exceptional curves over $\mathbb{R}$. When parameters $t$ occur, then a fundamental domain of these parameters is indicated. (For the determination of these parameter domains we refer to [57].) $M$ denotes the underlying tame bimodule. The pictures in the tables indicate which weight is inserted in which type of point(s) on the quotient of the Riemann sphere (with boundary). By 6.3.1 one has to check which automorphisms fix the given point(s). For the “classical” case, where $M = \mathbb{C} \oplus \mathbb{C}$, we refer to [71]; it is not treated again.

The letters $a, b, c, d, e$ in the tables indicate derived equivalence: Each tubular curve in the table with a letter from $a, b, c, d, e$ is derived equivalent to a curve in another table with the same letter.

In case $M = \mathbb{C} \oplus \overline{\mathbb{C}}$ it is easy to determine the geometric automorphism group $\text{Aut}(X)/\mathcal{G}$ from $\text{Aut}(X)$; the only ghost automorphism is induced by complex conjugation which is of order two.

From the tables we deduce the following

**Proposition A.1.1.** (1) There are no parameters in the domestic cases.
(2) If $X$ is tubular then Aut $X$ is finite.
### Table A.1. Domestic and tubular curves with $M = \mathbb{R} \oplus \mathbb{H}$

<table>
<thead>
<tr>
<th>Case</th>
<th>Symbol</th>
<th>Parameter</th>
<th>$\text{Aut} \mathcal{X}_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D 1</td>
<td>$(p \mid 2)$</td>
<td>$-$</td>
<td>$\mathbb{R}/2\pi \rtimes \mathbb{Z}_2$</td>
</tr>
<tr>
<td>T 1</td>
<td>$(2 2 \mid 2)$</td>
<td>$t \in [0, 1)$</td>
<td>$\mathbb{D}_4$, $t = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\mathbb{V}_4$, $t \neq 0$</td>
</tr>
</tbody>
</table>

### Table A.2. Domestic curves with $M = \mathbb{R} \oplus \mathbb{R}$ and $M = \mathbb{H} \oplus \mathbb{H}$

<table>
<thead>
<tr>
<th>Case</th>
<th>Weights</th>
<th>Symbol</th>
<th>$\mathbb{R} \oplus \mathbb{R}$</th>
<th>$\mathbb{H} \oplus \mathbb{H}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D 1</td>
<td>$p$</td>
<td>$(p)$</td>
<td>$\nabla \text{PGL}_2(\mathbb{R})$</td>
<td>$\nabla \text{PGL}_2(\mathbb{R})$</td>
</tr>
<tr>
<td>D 2</td>
<td>$(b \mid b)$</td>
<td>$(b \mid b)$</td>
<td>$-$</td>
<td>$\mathbb{C}^\ast /\mathbb{R}^\ast \rtimes \mathbb{Z}_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(b \mid b)$</td>
<td>$-$</td>
<td>$\mathbb{C}^\ast /\mathbb{R}^\ast \rtimes \mathbb{Z}_2$</td>
</tr>
<tr>
<td></td>
<td>$(p_1 p_2)$</td>
<td>$(p_1 p_2)$</td>
<td>$\mathbb{R}^\ast \times \mathbb{Z}_2$, $p_1 = p_2$</td>
<td>$\mathbb{R}^\ast \times \mathbb{Z}_2$, $p_1 = p_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(p_1 p_2)$</td>
<td>$\mathbb{R}^\ast$, $p_1 \neq p_2$</td>
<td>$\mathbb{R}^\ast$, $p_1 \neq p_2$</td>
</tr>
<tr>
<td>D 4</td>
<td>$(2 2 1)$</td>
<td>$(2 2 1)$</td>
<td>$-$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2 2 1)$</td>
<td>$-$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>D 5</td>
<td>$(2 1 2)$</td>
<td>$(2 1 2)$</td>
<td>$-$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2 1 2)$</td>
<td>$-$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>D 6</td>
<td>$(2 2 n)$</td>
<td>$(2 2 n)$</td>
<td>$\mathbb{Z}_2$, $n &gt; 2$</td>
<td>$\mathbb{Z}_2$, $n &gt; 2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2 2 n)$</td>
<td>$\mathbb{S}_3$, $n = 2$</td>
<td>$\mathbb{S}_3$, $n = 2$</td>
</tr>
<tr>
<td>D 7</td>
<td>$(2 3 3)$</td>
<td>$(2 3 3)$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>D 8</td>
<td>$(2 3 4)$</td>
<td>$(2 3 4)$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>D 9</td>
<td>$(2 3 5)$</td>
<td>$(2 3 5)$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
### Table A.3. Domestic curves with $M = \mathbb{C} \oplus \overline{\mathbb{C}}$

<table>
<thead>
<tr>
<th>Case</th>
<th>Weights</th>
<th>Symbol</th>
<th>$\text{Aut } \mathcal{X}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D 1</td>
<td>$p$</td>
<td>$(p)$</td>
<td>$\mathbb{R}_+ \times \mathbb{Z}_2$</td>
</tr>
<tr>
<td>D 2</td>
<td>$(\frac{p}{2})$</td>
<td></td>
<td>$\mathbb{V}_4$</td>
</tr>
<tr>
<td>D 3</td>
<td>$(\frac{p}{2})$</td>
<td></td>
<td>$\mathbb{V}_4$</td>
</tr>
<tr>
<td>D 4</td>
<td>$(p_1, p_2)$</td>
<td></td>
<td>$\mathbb{R}_+ \times \mathbb{V}<em>4$ $p_1 = p_2$ $\mathbb{R}</em>+ \times \mathbb{Z}_2$ $p_1 \neq p_2$</td>
</tr>
<tr>
<td>D 5</td>
<td>$(\frac{2}{2}, n)$</td>
<td>$\frac{1}{1}$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>D 6</td>
<td>$(\frac{2}{2}, 1)$</td>
<td>$\frac{1}{1}$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>D 7</td>
<td>$(\frac{2}{1}, \frac{3}{2})$</td>
<td></td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>D 8</td>
<td>$(\frac{2}{1}, \frac{3}{2})$</td>
<td></td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>Case</td>
<td>Weights</td>
<td>Symbol</td>
<td>$\mathbb{R} \oplus \mathbb{R}$</td>
</tr>
<tr>
<td>------</td>
<td>---------</td>
<td>--------</td>
<td>------------------</td>
</tr>
<tr>
<td>b T 1 a</td>
<td>$\begin{pmatrix} \frac{2}{2} &amp; \frac{2}{2} \ \frac{2}{2} &amp; \frac{2}{2} \end{pmatrix}$</td>
<td>$-\bigvee_4$</td>
<td>$t \in (0, 1)$</td>
</tr>
<tr>
<td>e T 2 d</td>
<td>$\begin{pmatrix} \frac{2}{1} &amp; \frac{4}{2} \ \frac{2}{1} &amp; \frac{4}{2} \end{pmatrix}$</td>
<td>$-\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>T 3</td>
<td>$\begin{pmatrix} \frac{3}{3} &amp; \frac{3}{2} \ \frac{3}{1} &amp; \frac{3}{2} \end{pmatrix}$</td>
<td>$-\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>T 4</td>
<td>$(2 3 6)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>T 5</td>
<td>$(2 4 4)$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>T 6</td>
<td>$(3 3 3)$</td>
<td>$\mathbb{S}_3$</td>
<td>$\mathbb{S}_3$</td>
</tr>
<tr>
<td>T 7</td>
<td>$\begin{pmatrix} \frac{1}{1} &amp; \frac{2}{2} \ \frac{1}{1} &amp; \frac{2}{2} \end{pmatrix}$</td>
<td>$-\mathbb{Z}_2$</td>
<td>$t = \pi/2$; $t \in (0, \pi)$</td>
</tr>
<tr>
<td>T 8</td>
<td>$(2 2 2)$</td>
<td>$\begin{pmatrix} \frac{1}{1} &amp; \frac{1}{1} &amp; \frac{2}{2} \ \frac{1}{1} &amp; \frac{1}{1} &amp; \frac{2}{2} \end{pmatrix}$</td>
<td>$\mathbb{A}_4$ $j = 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mathbb{D}_4$ $j = 1728$</td>
<td>$\mathbb{V}_4$ $j \neq 0, 1728$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mathbb{V}_4$ $j \neq 0, 1728$</td>
<td>$\mathbb{V}_4$ $j \neq 0, 1728$</td>
</tr>
</tbody>
</table>

**Table A.4.** Tubular curves for $M = \mathbb{R} \oplus \mathbb{R}$ and $M = \mathbb{H} \oplus \mathbb{H}$
### Table A.5. Tubular curves with $M = \mathbb{C} \oplus \overline{\mathbb{C}}$

<table>
<thead>
<tr>
<th>Case</th>
<th>Weights</th>
<th>Symbol</th>
<th>Parameter</th>
<th>Aut $X_\psi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T 1</td>
<td>[2 1 2]</td>
<td>$t \in (0, \pi)$</td>
<td>$\mathbb{V}_4$</td>
<td></td>
</tr>
<tr>
<td>T 2a</td>
<td>[2 2 2]</td>
<td>$t \in (0, 1)$</td>
<td>$\mathbb{V}_4$</td>
<td></td>
</tr>
<tr>
<td>T 3b</td>
<td>[2 2 2]</td>
<td>$t \in (0, 1)$</td>
<td>$\mathbb{V}_4$</td>
<td></td>
</tr>
<tr>
<td>T 4c</td>
<td>[2 2 2]</td>
<td>$t \in (0, 1]$</td>
<td>$\mathbb{V}_4$, $t = 1$, $t \neq 1$</td>
<td></td>
</tr>
<tr>
<td>T 5d</td>
<td>[2 4 2]</td>
<td>–</td>
<td>$\mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>T 6e</td>
<td>[2 1 2]</td>
<td>–</td>
<td>$\mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>T 7</td>
<td>[3 3 2]</td>
<td>–</td>
<td>$\mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>T 8</td>
<td>[3 3 2]</td>
<td>–</td>
<td>$\mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>T 9</td>
<td>[2 2 2]</td>
<td>–</td>
<td>$\mathbb{V}_4$</td>
<td></td>
</tr>
<tr>
<td>T 10</td>
<td>[2 2 2]</td>
<td>–</td>
<td>$\mathbb{V}_4$</td>
<td></td>
</tr>
</tbody>
</table>
## APPENDIX B

### The tubular symbols

<table>
<thead>
<tr>
<th>Table B.1. The 17 equivalence classes of tubular symbols</th>
</tr>
</thead>
</table>
| \[
\begin{array}{c}
(2 \ 4), \ (2 \ 2 \ 2) \\
(\begin{array}{c}
2 \\
4 \\
4 \\
\end{array}), \ (\begin{array}{c}
2 \\
2 \\
2 \\
\end{array}) \\
(3 \ 3), \ (3 \ 3) \\
\begin{array}{c}
2 \ 2 \\
1 \ 3 \\
2 \ 2 \\
1 \ 3 \\
\end{array}, \ (\begin{array}{c}
2 \ 2 \\
2 \ 2 \\
2 \ 2 \\
2 \ 2 \\
\end{array}) \\
(2 \ 2 \ 2 \ 2), \ (2 \ 2 \ 2 \ 2) \\
\end{array}
\] |
| \[
\begin{array}{c}
(2 \ 4 \ 1 \ 2), \ (2 \ 4 \ 1 \ 2) \\
(\begin{array}{c}
3 \ 3 \\
1 \ 2 \\
3 \ 3 \\
1 \ 2 \\
\end{array}), \ (\begin{array}{c}
3 \ 3 \\
1 \ 2 \\
3 \ 3 \\
1 \ 2 \\
\end{array}) \\
(2 \ 3 \ 6) \\
(2 \ 4 \ 4) \\
(3 \ 3 \ 3) \\
\begin{array}{c}
2 \ 2 \ 2 \\
1 \ 1 \ 2 \\
2 \ 2 \ 2 \\
1 \ 1 \ 2 \\
\end{array}, \ (\begin{array}{c}
2 \ 2 \ 2 \\
1 \ 1 \ 2 \\
2 \ 2 \ 2 \\
1 \ 1 \ 2 \\
\end{array}) \\
(2 \ 2 \ 2 \ 2) \\
\end{array}
\] |

See 0.4.5 for the definition of symbols. Two symbols are called equivalent if they yield the same Grothendieck group with Euler form. The 17 boxes show the 17 equivalence classes. We refer to [56] for details.
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