# A NOTE ON CONCEALED-CANONICAL ARTIN ALGEBRAS

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ABSTRACT. In this article some omnipresence condition is given which assures that a derived-canonical algebra is already concealed-canonical. The proof exploits the theory of coherent sheaves over exceptional curves.

## 1. INTRODUCTION

Throughout this article let $k$ be an arbitrary field, and $A$ be a finite dimensional $k$-algebra. We shall use the term module for a finitely generated right $A$-module. The category of (finitely generated right) $A$-modules is denoted by $\mathrm{mod}(A)$. Moreover, the derived category of bounded complexes of $A$-modules (see [4]) will be denoted by $\mathrm{D}^b(A)$. We call $A$ *derived-canonical*, if there is a canonical algebra $\Lambda$ (in the sense of Ringel/Crawley-Boevey [16]) such that $\mathrm{D}^b(A) \simeq \mathrm{D}^b(\Lambda)$ as triangulated categories. If moreover $\Lambda$ is of tubular type, then we call $A$ *derived-tubular*. Note that a derived-canonical algebra is connected since its derived category is. The Grothendieck group of $\mathrm{mod}(A)$ will be denoted by $\mathrm{K}_0(A)$, the Coxeter transformation on $\mathrm{K}_0(A)$ by $\Phi$.

Recall from [16] that for a canonical algebra $\Lambda$ the module category $\mathrm{mod}(\Lambda)$ is trisected into $\mathrm{mod}_+(\Lambda) \vee \mathrm{mod}_0(\Lambda) \vee \mathrm{mod}_-(\Lambda)$, where $\mathrm{mod}_0(\Lambda)$ is a stable separating tubular family, and there are no non-zero morphisms going from right to left. Recall from [11] that a $k$-algebra $A$ is called *concealed-canonical* (*almost concealed-canonical*, resp.), if for some canonical algebra $\Lambda$ there exists a tilting module lying in $\mathrm{mod}_+(\Lambda)$ (in $\mathrm{mod}_+(\Lambda) \vee \mathrm{mod}_0(\Lambda)$, resp.) and whose endomorphism algebra is isomorphic to $A$. If additionally $\Lambda$ is of tubular type, then we call $A$ a *tubular* algebra. Concealed-canonical algebras (in particular: tubular and canonical algebras) were studied by several authors (see for example [6, 9, 11, 13, 14, 16, 17], also [1, 2] and [5, 10, 15]).

It is well-known that the class of concealed-canonical algebras is not closed under derived equivalence. The aim of this note is to present a condition under which it follows that a derived-canonical algebra is concealed-canonical. The essential property will be the existence of some omnipresent indecomposable module. The notion of omnipresence was also successfully used in a similar context in [14, 17]. Recall that an $A$-module $M$ is called *omnipresent*, if each simple $A$-module occurs as a composition factor of $M$. Moreover, an Auslander-Reiten component is called *regular*, if it contains neither a projective nor an injective module, and it is called *semi-regular*, if it does not contain at the same time a projective and an injective module.

The main result of this note is the following

1991 *Mathematics Subject Classification.* 16G20, 16G70, 14H60, 18E30.

**Theorem.** *Let $A$ be a finite dimensional $k$-algebra over a field $k$. Then the following conditions* (1) *and* (2) *are equivalent*

(1) (a) *$A$ is derived-canonical, and*
    (b) *there is an omnipresent indecomposable $M \in \operatorname{mod}(A)$, such that*
        (i) *the class $[M] \in \mathrm{K}_0(A)$ has finite $\Phi$-period.*
        (ii) *$M$ lies in some regular Auslander-Reiten component in* $\operatorname{mod}(A)$.

(2) *$A$ is concealed-canonical.*

**Remarks.** (1) As the proof of the theorem will show, condition (b) can be replaced by the following condition:

(b′) There is a (finite) family of indecomposables $M_i \in \operatorname{mod}(A)$ $(i \in I)$ such that their direct sum is omnipresent, and such that all $M_i$ $(i \in I)$ lie in regular components in $\operatorname{mod}(A)$ and in the *same* tubular family in $\mathrm{D}^b(A)$.

(2) The *almost* concealed-canonical algebra $A$ over an algebraically closed field, which is given as path algebra of the quiver $1 \overset{x}{\underset{y}{\rightrightarrows}} 2 \xrightarrow{z} 3$ with relation $zx = 0$, shows, that in condition (ii) regularity cannot be replaced by semi-regularity. Namely, $A$ can be realized as endomorphism algebra of a tilting sheaf over the weighted projective line of weight type $(1, 2)$ (see [3]). The indecomposable projective $A$-module $P(3)$ is omnipresent, lying in a semi-regular tube of $A$.

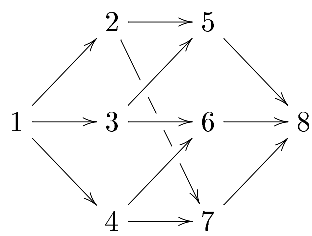
If we restrict to the tubular case we have a stronger result.

**Corollary.** *Let $A$ be a finite dimensional $k$-algebra over a field $k$. Then the following conditions are equivalent*

(1) *$A$ is derived-tubular, and there is an omnipresent indecomposable $M \in \operatorname{mod}(A)$ lying in some semi-regular Auslander-Reiten component in* $\operatorname{mod}(A)$.

(2) *$A$ is tubular.*

**Remark.** (3) Let $k$ be algebraically closed and $A$ be the poset algebra given by the quiver




with all 6 possible commutativity relations. Then $A$ is derived-canonical (of tubular type $(3, 3, 3)$), but not tubular (see [12]). The indecomposable projective injective $A$-module $P(8) = I(1)$ is omnipresent lying in a component in $\operatorname{mod}(A)$ which is not semi-regular. Thus, semi-regularity of the component in the corollary is indispensable.

Note, that in the theorem and in the corollary the implication (2) $\Longrightarrow$ (1) is trivial. In the proof of our result we shall use the coherent sheaves technique approach to the representation theory [3, 7]. This approach makes our proof rather simple. The following characterization of concealed-canonical algebras from [9] is of great importance for our proof: $A$ is concealed-canonical if and only if there exists an exceptional curve $\mathbb{X}$ (see [7]) – that is, a weighted projective line if $k$ is algebraically closed – and a torsion-free tilting object in the category $\operatorname{coh}(\mathbb{X})$ of coherent sheaves whose endomorphism algebra is isomorphic to $A$.

## 2. The derived category of a canonical algebra

Let $\Lambda$ be a canonical $k$-algebra over the field $k$ (compare [16]). By [16] $\operatorname{mod}(\Lambda)$ contains a stable separating tubular family $\operatorname{mod}_0(\Lambda)$, which is a coproduct of uniserial connected length categories $\mathcal{U}_x$ (called stable tubes). By the construction of [9] there is a small $k$-category $\mathcal{H}$, which is abelian, hereditary (that is, $\operatorname{Ext}^i_{\mathcal{H}}(-,-) = 0$ for all $i \geq 2$), noetherian, locally-finite (that is, all Hom and $\operatorname{Ext}^1$ spaces are of finite dimension over $k$), containing no non-zero projective object and admitting a torsion-free tilting object with endomorphism algebra isomorphic to $\Lambda$. Each indecomposable object in $\mathcal{H}$ is either in $\mathcal{H}_0$, the full subcategory of objects of finite length (so-called torsion objects), or in $\mathcal{H}_+$, the full subcategory formed by the torsion-free objects, which do not contain any non-zero torsion subobject. The relation $\operatorname{Hom}_{\mathcal{H}}(\mathcal{H}_0, \mathcal{H}_+) = 0$ holds. Moreover, $\mathcal{H}_0 = \operatorname{mod}_0(\Lambda)$.

There is an auto-equivalence $\tau : \mathcal{H} \longrightarrow \mathcal{H}$, called *Auslander-Reiten translation*, such that Serre duality holds naturally in $X$, $Y \in \mathcal{H}$:

$$\operatorname{Ext}^1_{\mathcal{H}}(X, Y) \simeq \mathrm{D}\operatorname{Hom}_{\mathcal{H}}(Y, \tau X),$$

where D denotes the duality $\operatorname{Hom}_k(-, k)$. Moreover, $\mathcal{H}$ admits almost split sequences, and for indecomposable end term $X$ in such a sequence the starting term is given by $\tau X$ (see [9, Thm. 6.1]).

The category $\mathcal{H}$ is also denoted by $\operatorname{coh}(\mathbb{X})$, and $\mathbb{X}$ equipped with $\operatorname{coh}(\mathbb{X})$ is called *exceptional curve* [7]. By tilting theory the categories $\operatorname{coh}(\mathbb{X})$ and $\operatorname{mod}(\Lambda)$ are derived-equivalent, $\mathrm{D}^b(\mathbb{X}) = \mathrm{D}^b(\Lambda)$, in particular also have isomorphic Grothendieck groups: $\mathrm{K}_0(\mathbb{X}) = \mathrm{K}_0(\Lambda)$. For each object $X$ in $\mathcal{H}$ denote by $[X]$ the class in $\mathrm{K}_0(\mathbb{X})$. We then have $[\tau X] = \Phi[X]$. Since $\mathcal{H}$ is hereditary, we have

$$\mathcal{D} := \mathrm{D}^b(\mathbb{X}) = \operatorname{add}\Bigg(\bigcup_{n \in \mathbb{Z}} \mathcal{H}[n]\Bigg),$$

where the $\mathcal{H}[n]$ are (disjoint) copies of $\mathcal{H}$; for each $X \in \mathcal{H}$ the copy in $\mathcal{H}[n]$ is denoted by $X[n]$. Each indecomposable object in $\mathcal{D}$ is of the form $X[n]$ for some (indecomposable) $X \in \mathcal{H}$ and some $n \in \mathbb{Z}$. For all $X$, $Y \in \mathcal{H}$ and all $m$, $n \in \mathbb{Z}$ we have

$$\operatorname{Hom}_{\mathcal{D}}(X[m], Y[n]) = \operatorname{Ext}^{n-m}_{\mathcal{H}}(X, Y);$$

in particular, if $m > n$ or $n > m + 1$, then $\operatorname{Hom}_{\mathcal{D}}(X[m], Y[n]) = 0$.

The Auslander-Reiten translation $\tau$ extends canonically to an auto-equivalence $\tau : \mathcal{D} \longrightarrow \mathcal{D}$ (which we denote by the same symbol).

## 3. Proof of the results

Assume that condition (1) from the theorem holds, and that $\mathrm{D}^b(A) = \mathrm{D}^b(\Lambda)$, where $\Lambda$ is canonical, and let $\mathbb{X}$ and $\mathcal{H}$ be as above. The proof has three steps:

*First step*: The omnipresent indecomposable $M \in \mathrm{mod}(A)$ lies in $\mathcal{H}_0[n]$ for some $n \in \mathbb{Z}$. Without loss of generality, we assume $n = 0$.

*Second step*: Realize $A$ as (endomorphism algebra of) a tilting complex $T$ in $\mathcal{D}$. By omnipresence, we immediately see that $T \in \mathcal{H}_0[-1] \cup \mathcal{H}$.

*Third step*: We have to show, that (using regularity) actually $T \in \mathcal{H}_+$, that is, $A$ can be realized as (endomorphism algebra of) a torsion-free tilting object in $A$ and hence is concealed-canonical (see [9]).

The *second* step is clear. For the *first*: We assume $M \in \mathcal{H}$. For non-tubular $\mathbb{X}$ and for non-zero $M \in \mathcal{H}_+$ it follows as in [8, Prop. 4.5], that $[M]$ has no finite $\Phi$-period. Thus, $M \in \mathcal{H}_0$, and $M$ lies in a stable tube $\mathcal{T}$ of finite rank. Observe, that in the tubular case, $M$ lies in a stable tube in any case (since $\mathrm{ind}\,\mathcal{H}$ consists entirely of stable tubes, compare [6]), not necessarily in $\mathcal{H}_0$, but after a possible change of the chosen separating tubular family $\mathrm{mod}_0(\Lambda)$ (and thus changing $\mathcal{H}$, compare [6, Prop. 7]) we can assume $M \in \mathcal{H}_0$.

It remains to prove the *third* step. We assume more generally, that $M$ lies in a semi-regular component $\mathcal{C}$ of $A$. Then $\mathcal{C}$ contains either no projective or no injective $A$-module.

*Case 1*. $\mathcal{C}$ contains no projective. Let $P$ be an indecomposable direct summand of the tilting complex $T$, which is an indecomposable projective $A = \mathrm{End}(T)$-module. Assume that $P \in \mathcal{H}_0$. By omnipresence, $\mathrm{Hom}_A(P, M) \neq 0$, and by orthogonality of the stable tubes, $P$ also lies in the tube $\mathcal{T}$. By assumption, $P$ and $M$ lie in different Auslander-Reiten components of $A$, therefore $\mathrm{Rad}_A^\infty(P, M) \neq 0$, and then also $\mathrm{Rad}_{\mathcal{D}}^\infty(P, M) \neq 0$, which gives a contradiction since $P$ and $M$ lie in the same stable tube $\mathcal{T}$, which is standard ([15]). Therefore, no indecomposable summand of $T$ lies in $\mathcal{H}_0$, hence $T \in \mathcal{H}_0[-1] \cup \mathcal{H}_+$ and therefore $A$ is dual to an almost concealed-canonical algebra.

*Case 2*. The component $\mathcal{C}$ contains no injective. Assume moreover, that there is an indecomposable projective $A$-module $P$ lying in $\mathcal{H}_0[-1]$. Then consider the corresponding injective $A$-module $I = \tau P[1]$. By omnipresence, $\mathrm{Hom}_A(M, I) \neq 0$, and by proceeding as above we see that $T \in \mathcal{H}_+ \cup \mathcal{H}_0$, and thus $A$ is almost concealed-canonical.

Now by [11], if $\mathcal{C}$ is regular, or if $\Lambda$ is of tubular type, it follows, that $A$ is concealed-canonical. This proves the theorem and the corollary.

## Acknowledgement

This note was written during a stay of the second named author at the University of Paderborn. He would like to express his gratitude to Helmut Lenzing for his hospitality. He also acknowledges a partial support of the Polish Scientific Grant KBN 2 PO3A 012 14. Both authors would like to thank Helmut Lenzing for stimulating discussions on the subject.

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