ON THE K-THEORY OF TUBULAR ALGEBRAS

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ABSTRACT. Let Λ be a tubular algebra over an arbitrary base field. We study the Grothendieck group $\mathrm{K}_0(\Lambda)$, endowed with the Euler form, and its automorphism group $\mathrm{Aut}(\mathrm{K}_0(\Lambda))$ on a purely K-theoretical level as in [7]. Our results serve as tools for classifying the separating tubular families of tubular algebras as in the example [5] and for determining the automorphism group $\mathrm{Aut}(\mathrm{D}^b(\Lambda))$ of the derived category of Λ.

1. Introduction

This article is concerned with the Grothendieck group (endowed with the Euler bilinear form) of a canonical algebra Λ over a non-algebraically closed field (as defined by Ringel and Crawley-Boevey [13]), in particular with those of tubular type (compare [7, 9]). All algebras which are derived-equivalent to Λ are also treated by our investigation, in particular tubular algebras (= concealed-canonical algebras ([9]) of tubular type) and derived-tubular algebras. This follows from the fact that an equivalence (of triangulated categories) of the derived categories induces an isomorphism of the Grothendieck groups preserving the Euler forms (compare [3]). The main aim of this paper is to develop the K-theoretical background which is needed to prove some results in the representation theory of tubular algebras and certain effects which occur when the base field is not algebraically closed.

A tubular algebra Λ admits a rational family of separating tubular families of stable tubes. Over an algebraically closed field all these stable separating tubular families for Λ are equivalent to each other as categories [12, 11]. This is not true in general over a non-algebraically closed field. In fact, in [5] we gave an example of a tubular canonical algebra over the real numbers which admits two equivalence classes of separating tubular families of stable tubes. The methods and results of the present paper allow to prove similar results for arbitrary tubular algebras.

Furthermore we give an example which shows that the distinction lemma in [1, 2] is not valid over non-algebraically closed fields.

We also determine the group of automorphisms of the Grothendieck group of Λ which preserve the Euler form. This is the first step of describing the automorphism group of the derived category, compare [10]. Starting point of our discussion is [7]. Furthermore, we correct some errors which appeared in that article. Some results in our paper are part of the authors doctoral thesis [6]. The author would like to thank Professor Helmut Lenzing for many helpful discussions.

2. Canonical bases and invariance of tubular symbols

We recall some definitions from [7]. A *bilinear group* is a finitely generated abelian group V equipped with a (non-symmetric) bilinear form

$$\langle -,-\rangle : V \times V \longrightarrow \mathbb{Z}$$

and an automorphism $\tau : V \longrightarrow V$ (called Coxeter transformation) such that for all $\mathbf{x}$, $\mathbf{y} \in V$ we have

$$\langle \mathbf{y}, \mathbf{x}\rangle = -\langle \mathbf{x}, \tau\mathbf{y}\rangle.$$

If additionally V is non-degenerate, then V is called *bilinear lattice*. We always assume that V is *normalized*, that is $\langle V, V\rangle = \mathbb{Z}$. Morphisms between bilinear groups are group homomorphisms which preserve the bilinear form and commute with the Coxeter transformation.

Let $V = (V, \langle -,-\rangle, \tau)$ be a bilinear group, and denote by $\operatorname{Kern} V$ the subgroup consisting of all $\mathbf{x} \in V$ such that $\langle \mathbf{x}, V\rangle = 0$ (equivalently, $\langle V, \mathbf{x}\rangle = 0$). We call a linear map $r : V \longrightarrow \mathbb{Z}$ a *rank* or *rank function*, if r is surjective and compatible with the Coxeter transformation, that is $r = r \circ \tau$. The *radical* of V is defined as $\operatorname{Rad} V = \{\mathbf{x} \in V \mid \tau\mathbf{x} = \mathbf{x}\}$. If $\mathbf{w}$ is in $\operatorname{Rad} V$ such that $\mathbf{w} \notin \operatorname{Kern} V$, and $c := [\mathbb{Z} : \langle V, \mathbf{w}\rangle]$, then $\operatorname{rk}_{\mathbf{w}} := \frac{1}{c}\langle -, \mathbf{w}\rangle$ defines a rank function, called $\mathbf{w}$-*rank*.

Assume now that V is a bilinear lattice. Direct summands of $\operatorname{Rad} V$ of rank 1 are called 1-tubes. Let r be a rank function. By scalar extension with $\mathbb{Q}$ we get $\mathbf{v} \in V$ which is generator of a 1-tube such that $r = \operatorname{rk}_{\mathbf{v}}$. Hence we have a bijection between rank functions and generators of 1-tubes.

Two ranks r and r' on V are called *similar*, if there is $\sigma \in \operatorname{Aut} V$ such that $r' = r\sigma$. If $\sigma \in \operatorname{Aut} V$ and $\mathbb{Z}\mathbf{w}$ and $\mathbb{Z}\mathbf{w}'$ are 1-tubes, then $\operatorname{rk}_{\mathbf{w}} = \operatorname{rk}_{\mathbf{w}'} \sigma$ if and only if $\sigma\mathbf{w}' = \mathbf{w}$. If V and V' are bilinear lattices and $\mathbf{w} \in V$ and $\mathbf{w}' \in V'$ are distinguished generators of 1-tubes, then an isomorphism between bilinear lattices $\sigma : V \longrightarrow V'$ is called *rank isomorphism* if $\sigma\mathbf{w} = \mathbf{w}'$. Denote by $\operatorname{Aut}_{\mathbf{w}} V$ the subgroup of $\operatorname{Aut} V$ consisting of automorphisms σ such that $\sigma(\mathbf{w}) = \mathbf{w}$.

Let V be a bilinear group. An element $\mathbf{u} \in V$ is called *root* if $\langle \mathbf{u}, \mathbf{u}\rangle > 0$ and $\frac{\langle \mathbf{u}, \mathbf{x}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle} \in \mathbb{Z}$ for all $\mathbf{x} \in V$. Let $\mathbf{u} \in V$ be a root with τ-period $p \geq 2$. We call

$$\mathbf{u}, \tau\mathbf{u}, \ldots, \tau^{p-1}\mathbf{u}$$

a *root basis*, if these elements are linearly independent over $\mathbb{Z}$ and

$$\langle \tau^i\mathbf{u}, \tau^j\mathbf{u}\rangle = \begin{cases} \langle \mathbf{u}, \mathbf{u}\rangle & j \equiv i \bmod p, \\ -\langle \mathbf{u}, \mathbf{u}\rangle & j \equiv i+1 \bmod p, \\ 0 & \text{else.} \end{cases}$$

A subgroup $T \subset V$ is called *p-tube* ($p \geq 2$), if it is generated by a root basis of length p. A p-tube T and a p'-tube T' are called *orthogonal* if $\langle T, T'\rangle = 0$.

Canonical (bilinear) lattices are defined in [7]. They serve as model for Grothendieck groups of canonical algebras (exceptional curves [8], resp.). The following proposition is a converse of [7, Prop. 7.7] and can be viewed as definition for canonical lattices. We omit the proof which is straightforward.

Proposition 2.1. *Let V be a (normalized) bilinear group and*

$B_{\mathbf{w}}$: $\quad \mathbf{a},\ \tau^j\mathbf{s_i}\ (1 \le i \le t,\ 0 \le j \le p_i - 2),\ \mathbf{w}$

a system of generators of V having the following properties (1)–(4):

(1) $\mathbf{w} \in \operatorname{Rad} V$, $\mathbf{w} \notin \operatorname{Kern} V$.

(2) $\mathbf{a}$ *is root of* $\mathbf{w}$*-rank* 1.

(3) *The* $\mathbf{s_i}$ *are roots of* $\mathbf{w}$*-rank* 0 *and their* τ*-orbits form root bases of pairwise orthogonal* p_i*-tubes.*

(4) $\langle \mathbf{a}, \mathbf{s_i}\rangle > 0$ *and* $\langle \mathbf{a}, \tau^j\mathbf{s_i}\rangle = 0$ *for* $0 < j \le p_i - 1$*; moreover,* $\langle \mathbf{a}, \mathbf{s_i}\rangle / \langle \mathbf{a}, \mathbf{w}\rangle \in \mathbb{Z}$.

Under these assumptions the following holds true: the numbers

$$\kappa := \langle \mathbf{a}, \mathbf{a}\rangle,\ \varepsilon := \frac{\langle \mathbf{a}, \mathbf{w}\rangle}{\langle \mathbf{a}, \mathbf{a}\rangle},\ e_i := \frac{\langle \mathbf{a}, \mathbf{s_i}\rangle}{\langle \mathbf{s_i}, \mathbf{s_i}\rangle},\ f_i := \frac{1}{\varepsilon}\frac{\langle \mathbf{a}, \mathbf{s_i}\rangle}{\langle \mathbf{a}, \mathbf{a}\rangle}$$

are positive integers, $B_{\mathbf{w}}$ *is a* $\mathbb{Z}$*-basis,* $\langle -, -\rangle$ *is non-degenerate,* $\varepsilon \in \{1, 2\}$*, and* $(V, \mathbf{w})$ *is a canonical bilinear lattice with symbol* (*compare* [7, Def. 7.6])

$$\sigma[V, \mathbf{w}] = \left(\begin{array}{c|c} p_1, \ldots, p_t & \\ d_1, \ldots, d_t & \varepsilon \\ f_1, \ldots, f_t & \end{array} \right), \tag{2.1}$$

where $d_i = e_i f_i$.

It is shown in [9] that the Grothendieck group of a concealed-canonical algebra has a basis as in the proposition (in particular this is true for a tubular algebra).

Let V be a canonical lattice as in the proposition. We call (contrary to [7]) the basis $B_{\mathbf{w}}$ *canonical* or $\mathbf{w}$*-canonical* and write it usually in the form

$$\mathbf{a} \mid \mathbf{s_1}, \tau\mathbf{s_1}, \ldots, \tau^{p_1-2}\mathbf{s_1} \mid \cdots \mid \mathbf{s_t}, \tau\mathbf{s_t}, \ldots, \tau^{p_t-2}\mathbf{s_t} \mid \mathbf{w} \tag{2.2}$$

We call the symbol (2.1) more precisely $\mathbf{w}$-symbol. A canonical lattice is thus a bilinear group admitting a canonical basis. Note, that the number κ as defined above can be calculated from the symbol, since it is the smallest positive integer such that $\kappa\frac{\varepsilon f_i}{e_i} \in \mathbb{Z}$ for $i = 1, \ldots, t$, see [7, Prop. 7.7]. Let $p = \operatorname{lcm}(p_1, \ldots, p_t)$ and

$$\delta[V] := p\Big(\sum_{i=1}^{t} e_i f_i\Big(1 - \frac{1}{p_i}\Big) - \frac{2}{\varepsilon}\Big).$$

Then V is called *domestic* (resp. *tubular*, *wild*) if $\delta[V] < 0$ ($= 0$, > 0, resp.) (compare [7] and also [9] for further characterizations).

We are interested in the question whether the numbers κ, ε, e_i, f_i (hence the symbol) are invariants of a canonical lattice. That is, given a canonical lattice and two canonical bases, are the symbols defined by these canonical bases (up to permutation) the same? This is not true in general, since there is a counterexample in the wild case (see Example 4.3), and also not true for some tubular cases, as we will see. But we show, that the symbol is an invariant of a *tubular* canonical lattice with respect to *rank* isomorphisms.

Let V be a bilinear lattice and B be a $\mathbb{Z}$-basis of V. We call the matrix C associated with the bilinear form relative to the basis B a *Cartan matrix*. If Φ is the matrix associated with the Coxeter transformation τ with respect to B, then we have the relation $\Phi = -C^{-1}C^t$ (see [7]).

Lemma 2.2. *Let $(V, \mathbf{w})$ be a canonical lattice. Under the notations above the numbers t, $\delta[V]$, the determinant of a Cartan matrix and the weights $p_1, \ldots, p_t$ (up to permutation) are invariants with respect to isomorphisms of bilinear groups.*

Moreover, the product $\kappa\varepsilon$ is an invariant with respect to rank isomorphism; it coincides with the index $[\mathbb{Z} : \langle V, \mathbf{w}\rangle]$.

Proof. The invariance of the weights follows from considering the Coxeter polynomial, see [7, Prop. 7.8]. In order to see the invariance of $\delta[V]$ distinguish the tubular from the non-tubular case. The tubular case is clear, since tubularity means that the radical of V is of rank two [7, 10.3]. If V is non-tubular, then by [7, 4.3+8.2] $\delta[V]$ is the unique non-zero integer δ, such that $\tau^p = \sigma_0^\delta$, where σ_0 is the shift automorphism associated to $\mathbf{w}$, and $\mathbf{w}$ generates the radical of V. $\square$

Let $(V, \mathbf{w})$ be a canonical lattice with basis as in Prop. 2.1. Let $\mathbf{u} \in V$ be a root of $\mathbf{w}$-rank 0. Define $e(\mathbf{u}) := \frac{\langle \mathbf{a}, \mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}$ and $f(\mathbf{u}) := \frac{1}{\varepsilon}\frac{\langle \mathbf{a}, \mathbf{u}\rangle}{\langle \mathbf{a}, \mathbf{a}\rangle}$. We call the fraction $\frac{e(\mathbf{u})}{f(\mathbf{u})} = \frac{\kappa\varepsilon}{\langle \mathbf{u}, \mathbf{u}\rangle}$ the *root quotient* of $\mathbf{u}$.

Lemma 2.3. *With the definitions from Prop. 2.1, let T_i be the tube generated by the τ-orbit of $\mathbf{s_i}$ $(i = 1, \ldots, t)$. Let $\mathbf{u} \in V$ be a root of $\mathbf{w}$-rank 0. Then there is $i \in \{1, \ldots, t\}$ and $n \in \mathbb{Z}$ such that $n\frac{e_i}{f_i} \in \mathbb{Z}$ and $\mathbf{u} = \mathbf{u}' + n\mathbf{w}$, where $\mathbf{u}'$ is a root in T_i; the root quotient of $\mathbf{u}$ is $\frac{e_i}{f_i}$. If, moreover, the τ-orbit of $\mathbf{u}$ forms a root basis, then there is some j such that (after possibly changing n) $\mathbf{u} = \pm\tau^j\mathbf{s_i} + n\mathbf{w}$.*

Proof. There is a representation $\mathbf{u} = \sum_{i=1}^t \mathbf{u_i} + n\mathbf{w}$ where $\mathbf{u_i} \in T_i$ and $n \in \mathbb{Z}$. Since $\mathbf{u}$ is a root, the number $\langle \mathbf{u}, \mathbf{u}\rangle = \sum_{i=1}^t \langle \mathbf{u_i}, \mathbf{u_i}\rangle$ divides all the non-negative integers $\langle \mathbf{u}, \mathbf{u_i}\rangle = \langle \mathbf{u_i}, \mathbf{u_i}\rangle$, hence $\mathbf{u} = \mathbf{u_i} + n\mathbf{w}$ for some i. Therefore, $\mathbf{u}' = \mathbf{u_i}$ is a root in T_i and hence also in V. Moreover, $\langle \mathbf{u}, \mathbf{u}\rangle = \langle \mathbf{u}', \mathbf{u}'\rangle = \langle \mathbf{s_i}, \mathbf{s_i}\rangle$ ([7, Prop. 5.2]). We get $\frac{e(\mathbf{u})}{f(\mathbf{u})} = \frac{\kappa\varepsilon}{\langle \mathbf{s_i}, \mathbf{s_i}\rangle} = \frac{e_i}{f_i}$. It is easy to check that if $\mathbf{v}$ is a root in T_i, then $\mathbf{v} + n\mathbf{w}$ is a root if and only if $n\frac{e_i}{f_i} \in \mathbb{Z}$. Moreover, if the τ-orbit of $\mathbf{u}$ forms a root basis, then also the τ-orbit of $\mathbf{u}'$ forms a root basis and hence $\mathbf{u}'$ is a root of length ± 1 (for the notion of length see [7]). $\square$

Theorem 2.4. *The symbol is a complete invariant of a tubular canonical lattice with respect to rank isomorphisms.*

Proof. The proof is based on the analysis of the list of tubular symbols in [7], see also Table 1. Tubular lattices which are rank isomorphic share the same sequence of weights and the same determinant of the Cartan matrices. In some cases we get pairs of symbols (Table 1), where these data coincide. In these cases the members of the pairs do not lead to rank isomorphic lattices since either the numbers $\kappa\varepsilon$ do not coincide, or (in the case $\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{pmatrix}$ and $(2\ 2 \mid 2)$) the root quotients do not coincide (which are $\frac{1}{2}$, 2 in the first and 1, 1 in the second case) (see also Remark 8.2 (3)).

To show completeness, let $(V, \mathbf{w})$ and $(V', \mathbf{w}')$ be tubular canonical lattices with the same symbols. Then we have canonical bases

$$B_{\mathbf{w}} : \qquad \mathbf{a},\ \tau^j\mathbf{s_i}\ (1 \le i \le t,\ 0 \le j \le p_i - 2),\ \mathbf{w}$$

of V and

$$B_{\mathbf{w}'}: \qquad \mathbf{a}',\ \tau^j\mathbf{s}_\mathbf{i}'\ (1 \le i \le t,\ 0 \le j \le p_i - 2),\ \mathbf{w}'$$

of V' as in Prop. 2.1. It is then possible to define a rank isomorphism σ on these bases in the obvious way which preserves the bilinear form since the symbols are the same. $\square$

With the same arguments one shows that the symbol is a complete invariant of a domestic canonical lattice with respect to isomorphisms.

3. Invariance of tubular decompositions

Lemma 3.1. *Let $(V, \mathbf{w})$ be a non-wild (that is, domestic or tubular) canonical lattice, and let $\sigma \in \operatorname{Aut}_\mathbf{w} V$. Let (2.2) be a canonical basis and T_i be the tube generated by the τ-orbit of $\mathbf{s}_\mathbf{i}$ $(i = 1, \dots, t)$. Then there is a permutation $\pi \in S_t$ such that for each $i \in \{1, \dots, t\}$ we have $\sigma(T_i) = T_{\pi(i)}$.*

Proof. By Lemma 2.3 there is a permutation $\pi \in S_t$ such that $\sigma(\mathbf{s}_\mathbf{i}) = \pm\tau^{k_i}\mathbf{s}_{\pi(i)} + n_i\mathbf{w}$, where $n_i \in \mathbb{Z}$ and $n_i\frac{e_{\pi(i)}}{f_{\pi(i)}} \in \mathbb{Z}$. Now $n_i \in \mathbb{Z}f_{\pi(i)}$ (which in case that the symbol is different from $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ easily follows from the fact that then each $e_i = 1$ or $f_i = 1$; applying σ to $\mathbf{s_1} + \tau\mathbf{s_1} = 2\mathbf{w}$ and involving $\sigma(\mathbf{w}) = \mathbf{w}$ shows the assertion also in that case). This implies $\sigma(\mathbf{s}_\mathbf{i}) \in T_{\pi(i)}$. $\square$

Example 3.2. The lemma is not true in general for wild canonical lattices. For example, consider the wild symbol $\begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$ (which is easily seen to be realizable as Grothendieck group of a canonical algebra over the real numbers $\mathbb{R}$), defined by the canonical basis $\mathbf{a} \mid \mathbf{s}, \tau\mathbf{s}, \tau^2\mathbf{s} \mid \mathbf{w}$. The canonical basis $\mathbf{a} - 3\mathbf{s} - 2\tau\mathbf{s} - \tau^2\mathbf{s} \mid -\mathbf{s} + \mathbf{w}, -\tau\mathbf{s} + \mathbf{w}, -\tau^2\mathbf{s} + \mathbf{w} \mid \mathbf{w}$ defines the same symbol. The tubes generated by the τ-orbits of $\mathbf{s}$ and $-\mathbf{s} + \mathbf{w}$, resp., are distinct.

Remark 3.3. The preceding lemma provides a proof (in the non-wild cases) of [7, Thm. 12.2]. Note that [7, Prop. 11.4] which is used there does not hold (even in the tubular case): for example, let $(V, \mathbf{w})$ be the tubular canonical lattice with canonical basis $\mathbf{a} \mid \mathbf{s_1} \mid \mathbf{w}$ and symbol $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. Then the τ-orbits of $\mathbf{s_1}$ and $\mathbf{s_1} + \mathbf{w}$, resp., generate two different tubes of elements of rank zero. Another obstruction is treated in section 7.

4. Further invariants

In this section we do not restrict to tubular canonical lattices. We show that the numbers κ and ε are invariants (up to rank isomorphism). By Lemma 2.2 we only know that the product $\kappa\varepsilon$ is an invariant.

In the canonical lattice $(V, \mathbf{w})$ we fix a canonical basis B as in (2.2) which defines a symbol (2.1) and assume that we have another $\mathbf{w}$-canonical basis $\widetilde{B}$. By

Lemma 2.3, after slightly changing $\widetilde{B}$ we get a $\mathbf{w}$-canonical basis B', yielding (up to permutation) the same symbol as $\widetilde{B}$, and which is of the form

$$B': \qquad \mathbf{a}' \mid \pm\tau^j \mathbf{s_i} + n_i\mathbf{w} \ (1 \le i \le t,\ 0 \le j \le p_i - 2) \mid \mathbf{w},$$

where $\mathbf{a}'$ has the shape

$$\mathbf{a}' = \mathbf{a} + \sum_{i=1}^{t}\sum_{j=0}^{p_i-2} \alpha_{ij}\tau^j \mathbf{s_i}.$$

Lemma 4.1. *Under the preceding assumptions we have*

$$\alpha_{ij} = \pm(p_i - 1 - j)n_i \frac{e_i}{f_i}$$

and

$$\langle \mathbf{a}', \mathbf{a}' \rangle = \langle \mathbf{a}, \mathbf{a} \rangle + \kappa\varepsilon \sum_{i=1}^{t} \pm(p_i - 1)n_i e_i + \kappa\varepsilon \sum_{i=1}^{t} n_i^2 \frac{e_i}{f_i}\frac{p_i(p_i-1)}{2}. \tag{4.1}$$

Proof. Exploit $\langle \mathbf{a}', \pm\tau^j\mathbf{s_i} + n_i\mathbf{w}\rangle = 0$ $(j = 1, \dots, p_i - 1)$. □

Proposition 4.2. *The numbers κ and ε are invariants of a canonical lattice $(V, \mathbf{w})$ with respect to rank isomorphisms.*

Proof. Denote $\kappa' = \langle \mathbf{a}', \mathbf{a}' \rangle$ and $\varepsilon' = \frac{\langle \mathbf{a}', \mathbf{w}\rangle}{\langle \mathbf{a}', \mathbf{a}'\rangle}$. The preceding lemma shows that $\langle \mathbf{a}, \mathbf{a} \rangle$ divides $\langle \mathbf{a}', \mathbf{a}' \rangle$. Since $\kappa\varepsilon = \kappa'\varepsilon'$, the formula also shows that $\langle \mathbf{a}', \mathbf{a}' \rangle$ divides $\langle \mathbf{a}, \mathbf{a} \rangle$, hence $\kappa = \kappa'$ and then also $\varepsilon = \varepsilon'$. □

Example 4.3. We show, that a $\mathbf{w}$-symbol in the wild case need not to be unique. Consider the canonical lattice $(V, \mathbf{w})$ with $\mathbf{w}$-canonical basis $\mathbf{a} \mid \mathbf{s_1} \mid \mathbf{s_2} \mid \mathbf{s_3} \mid \mathbf{s_4} \mid \mathbf{w}$, which defines the symbol $\left(\begin{array}{cccc|c} 2 & 2 & 2 & 2 & \\ 1 & 1 & 25 & 25 & \varepsilon \\ 1 & 1 & 5 & 5 & \end{array}\right)$. An easy calculation shows that

$$\mathbf{a} + \mathbf{s_1} + \mathbf{s_2} - \mathbf{s_3} \mid \mathbf{s_1} + \mathbf{w} \mid \mathbf{s_2} + \mathbf{w} \mid \mathbf{s_3} - \mathbf{w} \mid \mathbf{s_4} \mid \mathbf{w}$$

is also a $\mathbf{w}$-canonical basis which defines the symbol $\left(\begin{array}{cccc|c} 2 & 2 & 2 & 2 & \\ 9 & 9 & 9 & 25 & \varepsilon \\ 3 & 3 & 3 & 5 & \end{array}\right)$. This is a counterexample of the result [7, Thm. 13.1].

5. Slopes and rank functions

Let $(V, \mathbf{w})$ be a tubular canonical lattice with rank $\mathrm{rk} = \mathrm{rk}_{\mathbf{w}}$, let p be the least common multiple of the weights, and let $\mathbf{a} \in V$ be a root of rank 1 occurring in a $\mathbf{w}$-canonical basis.

Lemma 5.1. *Let $\mathbf{u} := \sum_{j=0}^{p-1} \tau^j \mathbf{a}$. Then $\mathbf{u}$, $\mathbf{w}$ forms a $\mathbb{Q}$-basis of $\mathbb{Q} \otimes \mathrm{Rad}\, V$.*

Proof. By [7, Prop. 10.3] $\mathrm{Rad}\, V$ is free of rank 2, hence $\mathbb{Q} \otimes \mathrm{Rad}\, V$ is two-dimensional over $\mathbb{Q}$. Since $\mathrm{rk}\,\mathbf{u} = p$ and $\mathrm{rk}\,\mathbf{w} = 0$ the elements $\mathbf{u}$ and $\mathbf{w}$ are linearly independent. □

Remark 5.2. In case $\left(\begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \,\middle|\, 2\right)$ the elements $\frac{1}{2}(\mathbf{u}-\mathbf{w})$, $\mathbf{w}$ form a $\mathbb{Z}$-basis of $\operatorname{Rad} V$. In all other tubular cases $\operatorname{Rad} V$ has a $\mathbb{Z}$-basis of the form $\frac{1}{c}\mathbf{u}$, $\mathbf{w}$, where $c \in \{1,2,3\}$. Compare the fourth column of Table 1.

Denote by $\overline{\mathbb{Q}}$ the disjoint union $\mathbb{Q} \cup \{\infty\}$, where ∞ will be considered as "fraction" $\frac{1}{0}$. Let $q \in \overline{\mathbb{Q}}$, $q = \frac{d}{r}$ such that $(d,r) = 1$, $r \geq 0$. Define $\widetilde{\mathbf{w}}_{\mathbf{q}} := r \cdot \mathbf{u} + d \cdot \mathbf{w}$ and $\mathbf{w}_{\mathbf{q}}$ such that $\mathbb{Z}\mathbf{w}_{\mathbf{q}}$ is a 1-tube and $\widetilde{\mathbf{w}}_{\mathbf{q}} \in \mathbb{N}\mathbf{w}_{\mathbf{q}}$.

Proposition 5.3. (1) *The* 1*-tubes of* $(V,\mathbf{w})$ *are exactly the* $\mathbb{Z}\mathbf{w}_{\mathbf{q}}$ $(q \in \overline{\mathbb{Q}})$.

(2) *The rank functions* (*up to sign*) *are in one to one correspondence with the elements* $q \in \overline{\mathbb{Q}}$.

Proof. It is not difficult to show that the map $q \mapsto \mathbb{Z}\mathbf{w}_{\mathbf{q}}$ is a bijection between $\overline{\mathbb{Q}}$ and 1-tubes. □

The automorphism group of $\operatorname{Rad} V$ can be identified with the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. By restriction each $\sigma \in \operatorname{Aut} V$ induces an element in Γ.

Denote by $\mathbb{P}(\operatorname{Rad} V)$ the set of all direct summands of rank 1 of $\operatorname{Rad} V$, which we also identify with $\overline{\mathbb{Q}}$. Each $\sigma \in \operatorname{Aut} V$ (or each $\sigma \in \Gamma$) induces a bijective map $\overline{\sigma}$ on $\mathbb{P}(\operatorname{Rad} V)$, which we consider as element of the projective modular group $\overline{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$. In this way the group $\operatorname{Aut} V$ acts on the set $\overline{\mathbb{Q}}$ (via $\sigma(\mathbf{w}_{\mathbf{q}}) = \pm\mathbf{w}_{\overline{\sigma}(q)}$). Let q, $q' \in \overline{\mathbb{Q}}$. We call q and q' *equivalent* if there is a $\sigma \in \operatorname{Aut} V$ such that $\sigma\mathbf{w}_{\mathbf{q}} = \mathbf{w}_{\mathbf{q}'}$. We call the classes of the induced equivalence relation on $\overline{\mathbb{Q}}$ *slope classes*.

6. Shift automorphisms

Let $(V,\mathbf{w})$ be a tubular canonical lattice with symbol (2.1) which is defined by a fixed $\mathbf{w}$-canonical basis (2.2) and let $p = \operatorname{lcm}(p_1,\ldots,p_t)$. For all $\mathbf{x}$, $\mathbf{y} \in V$ let

$$\langle\!\langle \mathbf{x},\mathbf{y} \rangle\!\rangle := \sum_{j=0}^{p-1} \langle \tau^j \mathbf{x}, \mathbf{y} \rangle.$$

Let $\mathrm{rk} = \mathrm{rk}_{\mathbf{w}}$, and define a degree function $\deg : V \longrightarrow \mathbb{Z}$ by

$$\deg(\mathbf{x}) = \frac{1}{\kappa\varepsilon} \langle\!\langle \mathbf{a},\mathbf{x} \rangle\!\rangle$$

for all $\mathbf{x} \in V$. Then we can define the *slope* of elements $\mathbf{x} \in V$, for which *not both* $\deg \mathbf{x}$ and $\mathrm{rk}\,\mathbf{x}$ are zero, by $\mu(\mathbf{x}) = \frac{\deg(\mathbf{x})}{\mathrm{rk}(\mathbf{x})} \in \overline{\mathbb{Q}}$. Obviously, $\mu(\mathbf{w}_{\mathbf{q}}) = q$ for all $q \in \overline{\mathbb{Q}}$. (The slope depends on the choice of deg (resp. $\mathbf{a}$) and rk (resp. $\mathbf{w}$).) We define certain shift automorphisms (as in [7]) and study their effect on the slope. For the general notion of a shift automorphism associated to an arbitrary tube we refer to [7].

Let

$$\sigma_0(\mathbf{x}) = \mathbf{x} - \frac{\langle \mathbf{w},\mathbf{x} \rangle}{\kappa\varepsilon}\mathbf{w}. \tag{6.1}$$

We have $\deg \sigma_0(\mathbf{x}) = \deg \mathbf{x} + p\,\mathrm{rk}\,\mathbf{x}$, $\mathrm{rk}\,\sigma_0(\mathbf{x}) = \mathrm{rk}\,\mathbf{x}$ and $\mu\sigma_0(\mathbf{x}) = \mu\mathbf{x} + p$.

For $i = 1, \dots, t$ let

$$\sigma_i(\mathbf{x}) = \mathbf{x} - \sum_{j=0}^{p_i-1} \frac{\langle \tau^j \mathbf{s_i}, \mathbf{x}\rangle}{\langle \mathbf{s_i}, \mathbf{s_i}\rangle} \tau^j \mathbf{s_i}. \tag{6.2}$$

We have $\deg \sigma_i(\mathbf{x}) = \deg \mathbf{x} + d_i \frac{p}{p_i} \operatorname{rk} \mathbf{x}$, $\operatorname{rk} \sigma_i(\mathbf{x}) = \operatorname{rk} \mathbf{x}$ and $\mu\sigma_i(\mathbf{x}) = \mu\mathbf{x} + d_i \frac{p}{p_i}$.

Remark. It is easy to see that the τ-orbit of $\mathbf{a}$ forms a root basis of a p-tube.

Let

$$\sigma_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} - \sum_{j=0}^{p-1} \frac{\langle \tau^j \mathbf{a}, \mathbf{x}\rangle}{\langle \mathbf{a}, \mathbf{a}\rangle} \tau^j \mathbf{a}. \tag{6.3}$$

We have $\deg \sigma_{\mathbf{a}}(\mathbf{x}) = \deg \mathbf{x}$, $\operatorname{rk} \sigma_{\mathbf{a}}(\mathbf{x}) = \operatorname{rk} \mathbf{x} - \varepsilon \deg \mathbf{x}$ and $\mu\sigma_{\mathbf{a}}(\mathbf{x}) = \frac{\mu\mathbf{x}}{1-\varepsilon\mu\mathbf{x}}$.

Each automorphism $\sigma \in \operatorname{Aut} V$ induces a bijective map $\overline{\sigma}$ from $\overline{\mathbb{Q}}$ into itself. We consider the subgroup G of $\overline{\Gamma}$ generated by these maps. Our aim is to calculate the (number of) orbits of the action of G on $\overline{\mathbb{Q}}$. Consider the subgroup S of G which is generated by the induced maps of σ_0, $\sigma_{\mathbf{a}}$ and all the σ_i ($i = 1, \dots, t$). Studying the list of tubular symbols in [7] (see also Table 1) we see that S is already generated by two of these maps, denoted by σ and ρ, where the following five cases can occur:

(1) $\sigma(q) = q + 1$, $\rho(q) = \frac{q}{1+q}$;
(2) $\sigma(q) = q + 2$, $\rho(q) = \frac{q}{1+q}$;
(3) $\sigma(q) = q + 3$, $\rho(q) = \frac{q}{1+q}$;
(4) $\sigma(q) = q + 1$, $\rho(q) = \frac{q}{1+2q}$;
(5) $\sigma(q) = q + 2$, $\rho(q) = \frac{q}{1+2q}$.

Lemma 6.1. *Let $S = \langle \sigma, \rho \rangle$. In each of the five cases the orbits of the action of S on $\overline{\mathbb{Q}}$ are the following*

1. $\overline{\mathbb{Q}}$
2. $\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \textit{ even}\}$, $\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \textit{ odd}\}$
3. $\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \equiv 0 \bmod 3\}$, $\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \not\equiv 0 \bmod 3\}$
4. $\{\frac{a}{b} \mid a \in \mathbb{N},\ b \in \mathbb{Z},\ b \textit{ even}\}$, $\{\frac{a}{b} \mid a \in \mathbb{N},\ b \in \mathbb{Z},\ b \textit{ odd}\}$
5. $\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \textit{ odd},\ b \textit{ odd}\}$, $\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \textit{ odd},\ b \textit{ even}\}$,
   $\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \textit{ even},\ b \textit{ odd}\}$.

Here, the notation $\frac{a}{b}$ tacitly means that a and b are coprime.

Proof. Denote by R and S the generators $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Γ, and by $\overline{R}$, $\overline{S}$ their images in $\overline{\Gamma}$, resp. It is sufficient to determine the number of orbits of the action of the subgroups $\langle \overline{R}, \overline{S} \rangle = \overline{\Gamma}$, $\langle \overline{R}, \overline{S}^2 \rangle$, $\langle \overline{R}, \overline{S}^3 \rangle$, $\langle \overline{R}^2, \overline{S} \rangle$, and $\langle \overline{R}^2, \overline{S}^2 \rangle$, resp., of $\overline{\Gamma}$ on $\overline{\mathbb{Q}}$, in other words the number of (equivalence classes of) cusps of these subgroups. For this see [4, III. §1]. (It is easily proved, that $\langle \overline{R}^2, \overline{S}^2 \rangle = \overline{\Gamma}(2)$ and $\langle \overline{R}, \overline{S}^3 \rangle = \overline{\Gamma}^1(3) = \overline{\Gamma}^0(3)$ in the notation of [4].) $\square$

7. Roots with no defined slope

Let $(V, \mathbf{w})$ be a tubular canonical lattice with canonical basis (2.2) and rank $\operatorname{rk} = \operatorname{rk}_{\mathbf{w}} = \frac{1}{\kappa\varepsilon}\langle -, \mathbf{w}\rangle$ and degree $\deg = \frac{1}{\kappa\varepsilon}\langle\langle \mathbf{a}, -\rangle\rangle$. Let $\mathbf{x} \in V$. We say that $\mathbf{x}$ *has a defined slope*, if $\operatorname{rk}\mathbf{x} \neq 0$ or $\deg \mathbf{x} \neq 0$. Otherwise we say that $\mathbf{x}$ *has no defined slope*. This definition is independent from the choice of our rank and degree, since by Lemma 5.1 it is easy to see, that $\mathbf{x}$ has no defined slope if and only if $\mathbf{x} \in (\operatorname{Rad} V)^{\perp}$ (that is, $\langle \mathbf{y}, \mathbf{x}\rangle = 0$ for all $\mathbf{y} \in \operatorname{Rad} V$). From this it also follows that if $\phi \in \operatorname{Aut} V$ then $\mathbf{x}$ has a defined slope if and only if $\phi\mathbf{x}$ has a defined slope.

Example 7.1. Assume that the symbol of $(V, \mathbf{w})$ is $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$, defined by the canonical basis $\mathbf{a} \mid \mathbf{s_1} \mid \mathbf{w}$. Then $\mathbf{x} := \mathbf{s_1} - \mathbf{w}$ is a root with $\operatorname{rk}\mathbf{x} = 0$ and $\deg \mathbf{x} = 0$, hence has no defined slope. If q is the quadratic form $q : V \longrightarrow \mathbb{Z}$ defined by $q(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v}\rangle$, then

$$\mathbf{x} \in q^{-1}(1) \cap (\operatorname{Rad} V)^{\perp}.$$

This is an example of a situation where the condition of the distinction lemma in [1, 2] is not fulfilled.

Lemma 7.2. *The tubular symbol* $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ *is the only one such that there exists a root which has no defined slope.*

Proof. By Lemma 2.3 each root of rank zero is of the form $\mathbf{x} = \pm \sum_{j=m}^{m+l} \tau^j \mathbf{s_i} + n\mathbf{w}$, where p_i does not divide $l+1$ and $n\frac{e_i}{f_i} \in \mathbb{Z}$. Then $\deg \mathbf{x} = (l+1) f_i \frac{p}{p_i} + np$ ($p = \operatorname{lcm}(p_1, \ldots, p_t)$). The proof of Lemma 3.1 shows that in all tubular cases different from $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ we have $n \in \mathbb{Z} f_i$. Therefore, $\deg \mathbf{x} = 0$ is only possible in this special case. □

8. Slope classes of tubular symbols

Recall that a (tubular) symbol is a scheme of natural numbers which is defined by a canonical basis of a (tubular) canonical lattice. We call two symbols *equivalent*, if it is possible to realize them by two canonical bases in the *same* canonical lattice. Let V be a tubular canonical lattice.

Theorem 8.1. *Table 1 shows the 17 equivalence classes of the tubular symbols. There are at most 2 slope classes, each lying dense in* $\overline{\mathbb{Q}}$*; the number of slope classes coincides with the number of symbols lying in one equivalence class. There is a subgroup* U *of* $\operatorname{Aut} V$*, generated by shift automorphisms associated to elements which are listed in the third column of the table, such that* U *acts transitively on the slope classes.*

Remarks 8.2. (1) The theorem, which will be proved in section 10, can be used to classify the separating tubular families over a tubular algebra similarly to the example described in [5]; this will be published in a forthcoming paper.

symbols	$\langle \mathbf{s_i}, \mathbf{s_i}\rangle$	gen. of U	rad.-basis	tH	$\mathbb{S}_{\sigma[V]}$	$\overline{U}$
$\begin{pmatrix}2\\4\end{pmatrix}$, $\left(\begin{matrix}2\\2\\2\end{matrix}\;\middle\vert\;2\right)$	1 4	$\mathbf{a}, \mathbf{w}$ $\mathbf{a}, \mathbf{s_1}, \mathbf{u}-2\mathbf{w}$	$\frac{1}{2}\mathbf{u}, \mathbf{w}$ $\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2$	1	$\overline{\Gamma}_2$
$\begin{pmatrix}2\\4\\2\end{pmatrix}$	1	$\mathbf{a}, \mathbf{w}, \frac{\mathbf{u}+2\mathbf{w}}{2}$	$\frac{1}{2}\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2$	1	$\overline{\Gamma}$
$\begin{pmatrix}2\\4\\4\end{pmatrix}$, $\left(\begin{matrix}2\\2\end{matrix}\;\middle\vert\;2\right)$	4 1	$\mathbf{a}, \mathbf{w}$ $\mathbf{s_1}, \frac{\mathbf{u}+\mathbf{w}}{2}$	$\mathbf{u}, \mathbf{w}$ $\frac{\mathbf{u}-\mathbf{w}}{2}, \mathbf{w}$	$\mathbb{Z}_2$	1	$\overline{\Gamma}_2$
$\begin{pmatrix}3\\3\end{pmatrix}$, $\begin{pmatrix}3\\3\\3\end{pmatrix}$	1 3	$\mathbf{a}, \mathbf{s_1}$	$\frac{1}{3}\mathbf{u}, \mathbf{w}$ $\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_3$	1	$\overline{\Gamma}_3$
$\begin{pmatrix}2&2\\1&3\end{pmatrix}$	3, 1	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2$	1	$\overline{\Gamma}$
$\begin{pmatrix}2&2\\1&3\\1&3\end{pmatrix}$	1, 3	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2$	1	$\overline{\Gamma}$
$\begin{pmatrix}2&2\\2&2\end{pmatrix}$, $\begin{pmatrix}2&2\\2&2\\2&2\end{pmatrix}$	1, 1 2, 2	$\mathbf{a}, \mathbf{s_1}$	$\frac{1}{2}\mathbf{u}, \mathbf{w}$ $\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\overline{\Gamma}_2$
$\begin{pmatrix}2&2\\2&2\\1&2\end{pmatrix}$, $(2\;2 \mid 2)$	1, 4 2, 2	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2\times\mathbb{Z}_2$ $\mathbb{Z}_2$	1 $\mathbb{Z}_2$	$\overline{\Gamma}_2$
$\begin{pmatrix}2&4\\1&2\end{pmatrix}$, $\begin{pmatrix}2&4\\1&2\\1&2\end{pmatrix}$	2, 1 1, 2	$\mathbf{a}, \mathbf{s_2}$	$\frac{1}{2}\mathbf{u}, \mathbf{w}$ $\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_4$	1	$\overline{\Gamma}_2$
$\begin{pmatrix}3&3\\1&2\end{pmatrix}$	2, 1	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_3$	1	$\overline{\Gamma}$
$\begin{pmatrix}3&3\\1&2\\1&2\end{pmatrix}$	1, 2	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_3$	1	$\overline{\Gamma}$
$(2\;3\;6)$	1, 1, 1	$\mathbf{a}, \mathbf{s_3}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2\times\mathbb{Z}_3$	1	$\overline{\Gamma}$
$(2\;4\;4)$	1, 1, 1	$\mathbf{a}, \mathbf{s_2}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2\times\mathbb{Z}_4$	$\mathbb{Z}_2$	$\overline{\Gamma}$
$(3\;3\;3)$	1, 1, 1	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_3\times\mathbb{Z}_3$	S_3	$\overline{\Gamma}$
$\begin{pmatrix}2&2&2\\1&1&2\end{pmatrix}$	2, 2, 1	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\overline{\Gamma}$
$\begin{pmatrix}2&2&2\\1&1&2\\1&1&2\end{pmatrix}$	1, 1, 2	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\overline{\Gamma}$
$(2\;2\;2\;2)$	1,1,1,1	$\mathbf{a}, \mathbf{s_1}$	$\mathbf{u}, \mathbf{w}$	$(\mathbb{Z}_2)^3$	S_4	$\overline{\Gamma}$

TABLE 1. Classes of tubular symbols

(2) It can be shown [6] that each tubular symbol can be realized as Grothendieck group of a tubular canonical algebra (or of a tubular exceptional curve) over

some field of characteristic zero. This can be done by inserting weights in suitable simple regular representations of suitable tame bimodules as described in [8].

(3) The second column of Table 1 shows the lists of numbers $\langle \mathbf{s_1}, \mathbf{s_1}\rangle, \dots, \langle \mathbf{s_t}, \mathbf{s_t}\rangle$. (For each pair of symbols, the upper (lower) list of numbers is associated to the left (right, resp.) symbol.) We see that two equivalent but different symbols can be distinguished by these lists.

Corollary 8.3. *Let U be the subgroup of* $\operatorname{Aut} V$ *as in Theorem 8.1. Then*

$$\{\overline{\sigma} \mid \sigma \in \operatorname{Aut} V\} = \{\overline{\sigma} \mid \sigma \in U\}.$$

The proof will be given in section 11.

9. The automorphism groups of tubular symbols

Denote by $\overline{U}$ the group as in Corollary 8.3, and denote by $\rho : \operatorname{Aut} V \longrightarrow \overline{U}$ the map $\sigma \mapsto \overline{\sigma}$. The group $\overline{U}$ can be considered as subgroup of $\overline{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$. More precisely, easy calculations (using the $\mathbb{Z}$-basis of $\operatorname{Rad} V$ given in Table 1, fourth column) show that $\overline{U}$ is of the form $\overline{\Gamma}$, $\overline{\Gamma}_2 := \langle \overline{R}, \overline{S}^2\rangle$ (or $\langle \overline{R}^2, \overline{S}\rangle$) or $\overline{\Gamma}_3 := \langle \overline{R}, \overline{S}^3\rangle$ (or $\langle \overline{R}^3, \overline{S}\rangle$), see Table 1. These groups coincide with the well-known (projective) congruence modular groups $\overline{\Gamma}$, $\overline{\Gamma}^0(2)$ (or $\overline{\Gamma}_0(2)$) and $\overline{\Gamma}^0(3)$ (or $\overline{\Gamma}_0(3)$) resp. (and hence are subgroups of $\overline{\Gamma}$ of index 1, 3 or 4, resp.), compare [4].

By [7, Prop. 12.1] the subgroup of $\operatorname{Aut} V$ which is generated by the shift automorphisms $\sigma_0, \sigma_1, \dots, \sigma_t$ from section 6 is isomorphic to the abelian group $\mathbb{L}(\mathbf{p}, \mathbf{d})$ on generators $\vec{x}_0, \dots, \vec{x}_t$ with relations $p_i \vec{x}_i = d_i \vec{x}_0$ for $1 \le i \le t$. As in [7] we denote by $\mathbb{S}_{\sigma[V]}$ the subgroup of the symmetric group $\mathbb{S}_t$ consisting of all permutations α preserving the symbol data, that is, satisfying $p_i = p_{\alpha(i)}$, $d_i = d_{\alpha(i)}$ and $f_i = f_{\alpha(i)}$ for all $i = 1, \dots, t$. Denote by $\langle -1\rangle$ the subgroup of $\operatorname{Aut} V$ generated by the negative identity which is of order 2.

Theorem 9.1. *Let V be a tubular canonical lattice and let U be the subgroup of* $\operatorname{Aut} V$ *as in Theorem 8.1. Let tH be the torsion group of the group $H = \mathbb{L}(\mathbf{p}, \mathbf{d})$. Then there is an exact sequence*

$$1 \longrightarrow \langle -1\rangle \cdot tH \cdot \mathbb{S}_{\sigma[V]} \longrightarrow \operatorname{Aut} V \xrightarrow{\ \rho\ } \overline{U} \longrightarrow 1. \tag{9.1}$$

Proof. It is sufficient to show that $\langle -1\rangle \cdot tH \cdot \mathbb{S}_{\sigma[V]}$ coincides with the kernel of ρ. This follows as in the proof of [7, Cor. 12.4] (compare Remark 3.3). □

The groups tH, $\mathbb{S}_{\sigma[V]}$ and $\overline{U}$ in each of the tubular cases are listed in Table 1.

10. Proof of Theorem 8.1

We treat each of the 23 tubular cases (which are listed in [7]), and fix a canonical basis (2.2). First we define two shift automorphisms such that the subgroup S of $\operatorname{Aut} V$ generated by them acts on $\overline{\mathbb{Q}}$ with the orbits which are given in Lemma 6.1. Then we show that either a) these orbits already coincide with the slope classes (by showing that the $\mathbf{w_q}$-symbols (q-symbols for short) for different orbits are distinct), or b) that we get the slope classes as orbits in $\overline{\mathbb{Q}}$ of the action of a subgroup S' of $\operatorname{Aut} V$ which arises from S by adding a further shift automorphism (by using then the same argument as in case a)). Of course, it is sufficient to determine q-symbols only for representatives q for each orbit. We get these q-symbols by calculating a $\mathbf{w_q}$-canonical basis B_q. (The ∞-symbol is always given by the given canonical basis (2.2).)

In the cases $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 \\ 1 & 3 \\ 1 & 3 \end{pmatrix}$, $\begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 3 & 3 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$, $(2\ 3\ 6)$, $(2\ 4\ 4)$, $(3\ 3\ 3)$, $\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ and $(2\ 2\ 2\ 2)$ the shifts associated to $\mathbf{a}$ and one of the $\mathbf{s_i}$ yield the case 1 from section 6. Hence there is only one slope class in these cases, namely $\overline{\mathbb{Q}}$.

10.1. The case $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Shifts at $\mathbf{a}$ and $\mathbf{w}$ yield the case 2 from section 6, hence there are at most 2 slope classes with representatives $q = 0$ and $q = \infty$, resp. For the representative $q = 0$ we have the canonical basis $B_0 :\ -\mathbf{s_1} \mid \tau\mathbf{a} \mid \frac{1}{2}\mathbf{u}$, and thus get the 0-symbol $\left(\begin{array}{c|c} 2 & \\ 2 & 2 \\ 2 & \end{array}\right)$.

By 2.4 there are exactly 2 slope classes, namely

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ even}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ odd}\}.$$

10.2. The case $\left(\begin{array}{c|c} 2 & \\ 2 & 2 \\ 2 & \end{array}\right)$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ (or $\mathbf{w}$) yield the case 5 from section 6, hence there are at most 3 slope classes with representatives $q = 0$, $q = 1$ and $q = \infty$, resp. The canonical basis $B_0 :\ -\mathbf{s_1} \mid \tau\mathbf{a} \mid \mathbf{u}$ gives the 0-symbol $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

The canonical basis $B_1 :\ \mathbf{a} \mid 2\mathbf{a} + \mathbf{w} \mid \mathbf{u} + \mathbf{w}$ gives the 1-symbol which coincides with the ∞-symbol.

Shift at the 1-tube which is generated by $\mathbf{u} - 2\mathbf{w}$ and defined by

$$\sigma_{\mathbf{u}-2\mathbf{w}}(\mathbf{x}) = \mathbf{x} - \frac{\langle \mathbf{u} - 2\mathbf{w}, \mathbf{x}\rangle}{4}(\mathbf{u} - 2\mathbf{w})$$

induces on the level of the slopes the map $q \mapsto \frac{3q+4}{-q-1}$ and therefore 1 and ∞ lie in the same slope class. Hence we have precisely 2 slope classes:

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ odd}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ even},\ b \text{ odd}\}.$$

10.3. The case $\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$. Shifts at $\mathbf{a}$ and $\mathbf{w}$ yield the case 2 from section 6, hence there are at most 2 slope classes with representatives $q = 0$ and $q = \infty$, resp. The canonical basis $B_0 :\ -\mathbf{s_1} \mid \tau\mathbf{a} \mid \frac{1}{2}\mathbf{u}$ gives the 0-symbol which coincides with the ∞-symbol.

Shift at the 1-tube which is generated by $\frac{1}{2}(\mathbf{u} + 2\mathbf{w})$ and defined by

$$\sigma_{\frac{1}{2}(\mathbf{u}+2\mathbf{w})}(\mathbf{x}) = \mathbf{x} - \frac{\langle \mathbf{u} + 2\mathbf{w}, \mathbf{x}\rangle}{4}(\mathbf{u} + 2\mathbf{w})$$

induces the map $q \mapsto \frac{4}{4-q}$, which shows that 0 and 1 (and hence ∞) lie in the same slope class. Hence there is exactly 1 slope class.

10.4. The case $\left(\begin{smallmatrix} 2\\ 4\\ 4\end{smallmatrix}\right)$. Shifts at $\mathbf{a}$ and $\mathbf{w}$ yield the case 2 from section 6, hence there are at most 2 slope classes with representatives $q=0$ and $q=\infty$, resp. The canonical basis $B_0:\ \mathbf{a}-\mathbf{w}\mid\mathbf{a}\mid\mathbf{u}$ gives the 0-symbol $\left(\begin{smallmatrix} 2\\ 2\end{smallmatrix}\,\middle|\, 2\right)$.

Hence there are precisely 2 slope classes:

$$\{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a \text{ even}\},\ \{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a \text{ odd}\}.$$

10.5. The case $\left(\begin{smallmatrix} 2\\ 2\end{smallmatrix}\,\middle|\, 2\right)$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ (or $\mathbf{w}$) yield the case 5 from section 6, hence there are at most 3 slope classes with representatives $q=0$, $q=-1$ and $q=\infty$, resp. The canonical basis $B_0:\ -\mathbf{s_1}\mid\tau\mathbf{a}\mid\mathbf{u}$ gives the 0-symbol which coincides with the ∞-symbol. The canonical basis $B_{-1}:\ \mathbf{s_1}\mid -2\mathbf{a}+4\mathbf{s_1}-\mathbf{w}\mid\frac{1}{2}(\mathbf{w}-\mathbf{u})$ gives the -1-symbol $\left(\begin{smallmatrix} 2\\ 4\\ 4\end{smallmatrix}\right)$.

Shift at the 1-tube which is generated by $\frac{1}{2}(\mathbf{u}+\mathbf{w})$ and defined by

$$\sigma_{\frac{1}{2}(\mathbf{u}+\mathbf{w})}(\mathbf{x})=\mathbf{x}-\frac{\langle\mathbf{u}+\mathbf{w},\mathbf{x}\rangle}{4}(\mathbf{u}+\mathbf{w}),$$

induces the map $q\mapsto\frac{1}{2-q}$, hence ∞ and 0 lie in the same slope class. Therefore we have exactly 2 slope classes:

$$\{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a,b \text{ odd}\},$$

$$\{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a \text{ even},\ b \text{ odd}\}\cup\{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a \text{ odd},\ b \text{ even}\}.$$

Furthermore, also the shifts at $\mathbf{s_1}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{w})$ yield these 2 slope classes. Note also that $\mu(\frac{1}{2}(\mathbf{u}+\mathbf{w}))=1$ and that the 1-symbol is $\left(\begin{smallmatrix} 2\\ 4\\ 4\end{smallmatrix}\right)$.

10.6. The case $\left(\begin{smallmatrix} 3\\ 3\end{smallmatrix}\right)$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ yield the case 3 from section 6, hence there are at most 2 slope classes with representatives $q=0$ and $q=\infty$, resp. The canonical basis $B_0:\ -\mathbf{s_1}\mid\tau\mathbf{a},\tau^2\mathbf{a}\mid\frac{1}{3}\mathbf{u}$ gives the 0-symbol $\left(\begin{smallmatrix} 3\\ 3\\ 3\end{smallmatrix}\right)$.

Hence we get precisely 2 slope classes:

$$\{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a\equiv 0 \bmod 3\},\ \{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a\not\equiv 0 \bmod 3\}.$$

10.7. The case $\left(\begin{smallmatrix} 3\\ 3\\ 3\end{smallmatrix}\right)$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ yield the case 3 from section 6, hence there are at most 2 slope classes with representatives $q=0$ and $q=\infty$, resp. The canonical basis $B_0:\ -\mathbf{s_1}\mid\tau\mathbf{a},\tau^2\mathbf{a}\mid\mathbf{u}$ gives the 0-symbol $\left(\begin{smallmatrix} 3\\ 3\end{smallmatrix}\right)$.

Hence there are exactly 2 slope classes:

$$\{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a\equiv 0 \bmod 3\},\ \{\frac{a}{b}\mid a\in\mathbb{Z},\ b\in\mathbb{N},\ a\not\equiv 0 \bmod 3\}.$$

10.8. The case $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ yield the case 2 from section 6, hence there are at most 2 slope classes with representatives $q = 0$ and $q = \infty$, resp. The canonical basis $B_0 : \ -\mathbf{s_1} \mid \tau\mathbf{a} \mid \tau\mathbf{a} - 2\tau\mathbf{s_2} + \mathbf{w} \mid \frac{1}{2}\mathbf{u}$ gives the 0-symbol $\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}$. Therefore we have precisely 2 slope classes:

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ even}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ odd}\}.$$

10.9. The case $\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ yield the case 2 from section 6, hence there are at most 2 slope classes with representatives $q = 0$ and $q = \infty$, resp. The canonical basis $B_0 : \ \mathbf{s_1} - 2\mathbf{w} \mid \mathbf{a} \mid \mathbf{a} - \mathbf{s_2} + \mathbf{w} \mid \mathbf{u}$ gives the 0-symbol $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Hence there are exactly 2 slope classes, namely

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ even}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ odd}\}.$$

10.10. The case $\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{pmatrix}$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ yield the case 2 from section 6, therefore we have at most 2 slope classes with representatives $q = 0$ and $q = \infty$, resp. The canonical basis $B_0 : \ \mathbf{s_1} - \mathbf{w} \mid \mathbf{a} - \mathbf{s_2} + \mathbf{w} \mid \mathbf{a} \mid \mathbf{u}$ gives the 0-symbol $(2\ 2 \mid 2)$. Hence we have exactly 2 slope classes:

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ even}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ odd}\}.$$

10.11. The case $(2\ 2 \mid 2)$. Shifts at $\mathbf{a}$ and $\mathbf{s_1}$ yield the case 4 from section 6, hence there are at most 2 slope classes with representatives $q = 0$ and $q = \infty$, resp. The canonical basis $B_0 : \ -\mathbf{s_2} \mid \tau\mathbf{a} \mid 2\tau\mathbf{a} - 2\tau\mathbf{s_1} + \mathbf{w} \mid \mathbf{u}$ gives the 0-symbol $\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{pmatrix}$. Hence there are precisely 2 slope classes:

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ b \text{ even}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ b \text{ odd}\}.$$

10.12. The case $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$. Shifts at $\mathbf{a}$ and $\mathbf{s_2}$ yield the case 2 from section 6, therefore we have at most 2 slope classes with representatives $q = 0$ and $q = \infty$, resp. The canonical basis $B_0 : \ -\mathbf{s_2} \mid \mathbf{a} - 2\mathbf{s_2} - \tau\mathbf{s_2} - \tau^2\mathbf{s_2} + \mathbf{w} \mid \tau\mathbf{a}, \tau^2\mathbf{a}, \tau^3\mathbf{a} \mid \frac{1}{2}\mathbf{u}$ gives the 0-symbol $\begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$. Hence there are exactly 2 slope classes:

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ even}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ odd}\}.$$

10.13. The case $\begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$. Shifts at $\mathbf{a}$ and $\mathbf{s_2}$ yield the case 2 from section 6, hence there are at most 2 slope classes with representatives $q = 0$ and $q = \infty$.

The canonical basis B_0 : $-\tau^3\mathbf{s_2} \mid 2\mathbf{a} - 2\mathbf{s_1} - \mathbf{s_2} - \tau\mathbf{s_2} + 2\mathbf{w} \mid \mathbf{a}, \tau\mathbf{a}, \tau^2\mathbf{a} \mid \mathbf{u}$ gives the 0-symbol $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$. We get precisely 2 slope classes:

$$\{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ even}\},\ \{\frac{a}{b} \mid a \in \mathbb{Z},\ b \in \mathbb{N},\ a \text{ odd}\}.$$

11. Proof of Corollary 8.3

Let $\sigma \in \operatorname{Aut} V$, and let $q \in \overline{\mathbb{Q}}$ such that $\overline{\sigma}(0) = q$. Since U acts transitively on the slope classes, there is $u \in U$ such that $\overline{u}(q) = 0$. We shall show that $\overline{u\sigma} = \overline{\sigma_1}$ for some $\sigma_1 \in U$, which then will prove the corollary. Since $\overline{u\sigma}(0) = 0$, the element $\overline{u\sigma}$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ with $c \in \mathbb{Z}$. As the analysis in section 10 shows, in each tubular case one of the five cases from section 6 applies, and the assertion is clear whenever the case 1, 2 or 3 from section 6 applies. Thus it remains to show the assertion for the cases 10.2, 10.5 and 10.11. Taking into account that the element $\overline{u\sigma}$ acts on each slope class one easily sees that $c \in 2\mathbb{Z}$ in these remaining cases, and then the assertion is also clear.

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