THE LIE ALGEBRA ASSOCIATED TO A UNIT FORM

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Abstract. To any unit form $q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} q_{ij}x_ix_j$, $q_{ij} \in \mathbb{Z}$, we associate a Lie algebra $\tilde{\mathrm{G}}(q)$ —an intersection matrix Lie algebra in the terminology of Slodowy— by means of generalized Serre relations. For a non-negative unit form the isomorphism type of $\tilde{\mathrm{G}}(q)$ is determined by the equivalence class of q. Moreover for q non-negative and connected with radical of rank zero or one respectively, the algebras $\tilde{\mathrm{G}}(q)$ turn out to be exactly the simply-laced Lie algebras which are finite-dimensional simple or affine Kac-Moody, respectively. In case q is connected, non-negative of corank two and not of Dynkin type $\mathbb{A}_n$, the algebra $G(q)$ is elliptic.

1. Introduction

We recall that a *unit form* is a quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$, $q(x) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} q_{ij}x_ix_j$, with integer coefficients $q_{ij} \in \mathbb{Z}$. Each unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ has an associated *Cartan matrix* C given by $C_{ij} = q(c_i+c_j)-q(c_i)-q(c_j)$, where $c_1, \dots, c_n$ is the canonical basis of $\mathbb{Z}^n$. To any unit form $q : \mathbb{Z}^n \to \mathbb{Z}$ we attach a $\mathbb{Z}^n$-graded complex Lie algebra $\mathrm{G}(q)$ with generators e_i, e_{-i}, h_i $(1 \le i \le n)$ which are homogeneous of degree c_i, $-c_i$ and 0, respectively, and subject to the following relations:

(R1) $[h_i, h_j] = 0$, for all i and j,
(R2) $[h_i, e_{\varepsilon j}] = \varepsilon C_{ij} e_{\varepsilon j}$, for all i, j and $\varepsilon = \pm 1$,
(R3) $[e_{\varepsilon i}, e_{-\varepsilon i}] = \varepsilon h_i$, for all i and $\varepsilon = \pm 1$,
(R∞) $[e_{\varepsilon_1 i_1}, \dots, e_{\varepsilon_t i_t}] = 0$, whenever $q(\sum_{j=1}^{t} \varepsilon_j c_{i_j}) > 1$ for $\varepsilon_j = \pm 1$.

For the relations (R∞) we use *multibrackets*, defined inductively by $[x_1, x_2, \dots, x_t] = [x_1, [x_2, \dots, x_t]]$. Clearly, the usual *Serre relations*,

(R4) $(\mathrm{ad}\ e_{\varepsilon i})^{1+n}(e_{\delta j}) = 0$, where $n = \max\{0, -\varepsilon\delta C_{ij}\}$, for $\varepsilon, \delta = \pm 1$ and $1 \le i, j \le n$,

are a special case of the infinite set of relations (R∞).

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In section 5, we will extend $\mathrm{G}(q)$ by the $\mathbb{C}$-dual of the radical of q to $\tilde{\mathrm{G}}(q)$, again a $\mathbb{Z}^n$-graded Lie algebra. We note that $\tilde{\mathrm{G}}(q)$ agrees with $\mathrm{G}(q)$ if q is positive definite.

The algebras $\tilde{\mathrm{G}}(q)$ associated to a unit form q were studied by Slodowy in [18], see also [19], under the name intersection matrix Lie algebras. One aim of the present paper is to connect this concept with recent work on quadratic forms linked to representation theory of finite-dimensional algebras, see for example [4, 7].

While in general passage to an equivalent form $q' = q \circ T$, for an automorphism T of $\mathbb{Z}^n$, will not preserve unit forms, the class of unit forms is stable under Gabrielov transformations, hence under Gabrielov equivalence (G-equivalence, for short). A unit form is called *non-negative* if $q(x) \geq 0$ for each $x \in \mathbb{Z}^n$. Moreover, we associate with q a bigraph, which has vertices $1, \ldots, n$ (as many as q has variables) and $|q_{ij}|$ solid (resp. dotted) lines between i and j if $q_{ij} < 0$ (resp. $q_{ij} > 0$) and say that q is *connected* if this bigraph is connected.

Theorem 1.1. *If q and q' are G-equivalent then $\mathrm{G}(q)$ and $\mathrm{G}(q')$ are isomorphic as graded Lie algebras.*

We provide a proof of Theorem 1.1 in the language of unit forms. In a slightly different setting a proof is also given in Slodowy's habilitation thesis [18] which, however, is not easily accessible.

In general, equivalent unit forms are not G-equivalent, see Remark 4.1. The hypothesis of q and q' being G-equivalent is thus usually difficult to verify. However, our investigation mainly concerns non-negative unit forms. Such a form is determined up to equivalence by its *Dynkin type*, which is a disjoint union of Dynkin diagrams, and its *corank* r, defined as the rank of the radical of q, see [1] (also [3] for the case of corank zero). In that case, we have the following characterization of G-equivalence.

Proposition 1.2. *Two connected, non-negative unit forms q and q' are equivalent if and only if they are G-equivalent.*

This result will be used to prove Theorem 1.3.

Theorem 1.3. *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected non-negative unit form. Let $r = \mathrm{rank}(\mathrm{rad}\, q)$ and Δ its Dynkin type.*

(a) *If $r = 0$, that is, q is positive definite, then the algebras $\mathrm{G}(q) = \tilde{\mathrm{G}}(q)$ are exactly the simply-laced finite-dimensional simple Lie algebras.*

(b) *If $r = 1$, then the algebras $\tilde{\mathrm{G}}(q)$ are exactly the simply-laced affine Kac-Moody Lie algebras.*

(c) *Let $r = 2$. Then we have:*

  (i) *If $\Delta = \mathbb{D}_n$ ($n \geq 4$) or $\Delta = \mathbb{E}_n$ ($n = 6, 7, 8$), then the algebras $\tilde{\mathrm{G}}(q)$ are exactly the Lie algebras associated to a simply-laced elliptic root system $\Gamma(R, G)$ with $\Delta(R) = \Delta$.*

  (ii) *If $\Delta = \mathbb{A}_n$ ($n \geq 2$), then the Lie algebra associated to a simply-laced elliptic root system $\Gamma(R, G)$ with $\Delta(R) = \Delta$ is a quotient of $\tilde{\mathrm{G}}(q)$.*

Finally, we show in Proposition 6.6, that in the three cases (a), (b) and (c)(i), *finitely many* relations from (R1)$-$(R∞) are sufficient for the definition of $\mathrm{G}(q)$. The fact that the assertion in case (c)(ii) is weaker will be discussed in more detail in Remark 6.4.

For standard information on elliptic Lie algebras we refer to the article [15] by Saito and Yoshii. For a significant subclass, additional information is provided by Lin and Peng [10], respectively Schiffmann [16], applying a variant of Ringel's Hall algebra approach [14, 5] to the module category over a tubular algebra [13], respectively the category of coherent sheaves over a weighted projective line of tubular type [11].

2. Relationship to a construction of Borcherds

Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a unit form. The Lie algebra $\mathrm{G}(q)$ associated to q is the quotient of the free Lie algebra in the generators e_i, e_{-i}, h_i ($1 \leq i \leq n$) by the ideal generated by the relations (R1), (R2), (R3) and (R∞).

We note that $\mathrm{G}(q)$ is a $\mathbb{Z}^n$-graded Lie algebra where the grading is given by $\deg(e_i) = c_i$, $\deg(e_{-i}) = -c_i$ and $\deg(h_i) = 0$. In general, when we consider a *morphism of Lie algebras graded over* $\mathbb{Z}^n$, we mean by that a morphism of Lie algebras $\varphi : G \to H$ such that there exists a linear map $\Phi : \mathbb{Z}^n \to \mathbb{Z}^n$ satisfying $\varphi(G_\alpha) \subseteq H_{\Phi(\alpha)}$. Each *monomial*, that is, an element obtained from the generators using iteratively the bracket only, has a well defined degree. The monomials of the degree α form the subspace $\mathrm{G}(q)_\alpha$. Notice, that $\mathrm{G}(q)_\alpha$ is generated by the multibrackets of degree α.

Lemma 2.1. *In $\mathrm{G}(q)$ we have for $\varepsilon_j = \pm 1$ and $1 \leq i_j \leq n$,*

$$[h_k, e_{\varepsilon_1 i_1}, \ldots, e_{\varepsilon_t i_t}] = \sum_{j=1}^{t} \varepsilon_j C_{k i_j} [e_{\varepsilon_1 i_1}, \ldots, e_{\varepsilon_t i_t}]$$

and for each non-zero vector α of $\mathbb{Z}^n$, the vector space $\mathrm{G}(q)_\alpha$ *is generated by all expressions* $[e_{\varepsilon_1 i_1}, \ldots, e_{\varepsilon_t i_t}]$ *with* $\sum_{j=1}^t \varepsilon_j c_{i_j} = \alpha$.

Proof. The first formula follows easily by induction using (R2). The second part follows from the fact that $\mathrm{G}(q)_\alpha$ is generated by the multibrackets of degree α. □

Define $R^1 = q^{-1}(1)$, $R^0 = q^{-1}(0)$ and $R = R^0 \cup R^1$. Further set $N_\alpha = \mathbb{C}E_\alpha$ for $\alpha \in R^1$, $N_\alpha = \mathbb{C}^n/\mathbb{C}\alpha$ for $\alpha \in R^0$ and let $\pi_\alpha : \mathbb{C}^n \to N_\alpha$ be the canonical projection for any $\alpha \in R^0$. Choose a non-symmetric bilinear form $B : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ such that $q(x) = B(x,x)$ for all $x \in \mathbb{Z}^n$ and set $\epsilon(\alpha,\beta) = (-1)^{B(\alpha,\beta)}$. Furthermore, let $q(-,-) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be the $\mathbb{C}$-bilinear form which satisfies $q(\alpha|\beta) = B(\alpha,\beta) + B(\beta,\alpha)$ for all $\alpha, \beta \in \mathbb{Z}^n$. Now, let $N = \bigoplus_{\alpha \in R} N_\alpha$ and define the following bracket rules, which depend on the choice of B:

For $\alpha, \beta \in R^0$, $\gamma, \delta \in R^1$ and $f, g \in \mathbb{C}^n$, let

$$\text{(S1)} \qquad [\pi_\alpha(f), \pi_\beta(g)] = \epsilon(\alpha,\beta) q(f|g) \pi_{\alpha+\beta}(\alpha)$$

$$\text{(S2)} \qquad [\pi_\alpha(f), E_\delta] = \epsilon(\alpha,\delta) q(f|\delta) E_{\alpha+\delta}$$

$$\text{(S3)} \qquad [E_\gamma, E_\delta] = \begin{cases} \epsilon(\gamma,\delta) E_{\gamma+\delta}, & \text{if } \gamma+\delta \in R^1 \\ \epsilon(\gamma,\delta) \pi_{\gamma+\delta}(\gamma), & \text{if } \gamma+\delta \in R^0 \\ 0, & \text{else.} \end{cases}$$

Proposition 2.2. *If q is a connected, non-negative unit form, then the bracket rules above define a graded Lie algebra structure on the space N. Moreover, N is independent of the choice of B and there is a surjective homomorphism of graded Lie algebras* $\mathrm{G}(q) \to N$.

Proof. The first assertion follows by a lengthy calculation, or by observing that N is, up to a slight modification, the Lie algebra constructed with the vertex algebra approach by Borcherds in [2], see also [15]. (The modification concerns to extend N by the radical of q, analogously as this is done in section 5 with $\mathrm{G}(q)$.)

Let B' be any bilinear form with $q(x) = B'(x,x)$, define $\epsilon'(\alpha,\beta) = (-1)^{B'(\alpha,\beta)}$ and denote by N' the corresponding graded Lie algebra with typical elements E'_α, $\pi'_\alpha(h)$.

Set $\sigma(\alpha) = \prod_{i<j} (\epsilon(c_i,c_j)\epsilon'(c_i,c_j))^{\alpha_i \alpha_j}$ and define $\varphi : N \to N'$ linearly by $\varphi(E_\alpha) := \sigma(\alpha) E'_\alpha$ for any $\alpha \in R^1$ and $\varphi(\pi_\alpha(h)) = \sigma(\alpha)\pi'_\alpha(h)$ for any $\alpha \in R^0$ and any $h \in \mathbb{C}^n$.

We have $\sigma(\alpha)\sigma(\beta)\sigma(\alpha+\beta) = \prod_{i<j} (\epsilon(c_i,c_j)\epsilon'(c_i,c_j))^{\alpha_i\beta_j + \beta_i\alpha_j}$, since $\epsilon(c_i,c_j)^{2\alpha_i\alpha_j + 2\beta_i\beta_j} = 0$. Similarly, $\prod_{i<j} \epsilon(c_i,c_j)^{\alpha_i\beta_j+\beta_i\alpha_j} = (-1)^{B(\alpha,\beta)}$

since $(-1)^{B(c_i,c_i)\cdot 2\alpha_i\beta_i} = 0$. Hence $\sigma(\alpha)\sigma(\beta)\sigma(\alpha+\beta) = \epsilon(\alpha,\beta)\epsilon'(\alpha,\beta)$, or equivalently $\sigma(\alpha)\sigma(\beta)\epsilon'(\alpha,\beta) = \epsilon(\alpha,\beta)\sigma(\alpha+\beta)$, from which it easily follows that φ is a homomorphism of graded Lie algebras. The bijectivity of φ is obvious. Hence, N is independent of the choice of B.

It is further easy to check that the elements E_{c_i}, $-E_{-c_i}$ $(1 \le i \le n)$ together with $H_i = [E_{c_i}, -E_{-c_i}]$ satisfy the relations (R1)$-$(R∞). Thus there is a homomorphism of graded Lie algebras $\mathrm{G}(q) \to N$ mapping $e_{\varepsilon i}$ to $\varepsilon E_{\varepsilon c_i}$ and h_i to H_i. In order to prove that this homomorphism is surjective, we show that the Lie algebra $\hat{N} \subset N$ generated by the elements E_{c_i}, E_{-c_i} $(1 \le i \le n)$ coincides with N.

We first show, that for any $\alpha = \sum_{i=1}^n \alpha_i c_i \in R^1$ with $|\alpha| = \sum_{i=1}^n |\alpha_i| > 1$, there exists a $\beta \in R$ with $|\beta| < |\alpha|$ and $\alpha - \beta = \varepsilon c_i$ for some i and some $\varepsilon = \pm 1$.

Let ε_i be the sign of α_i. Suppose that for any i with $\alpha_i \neq 0$, we have $2 \le q(\alpha - \varepsilon_i c_i)$. Then $q(\alpha|\varepsilon_i c_i) \le 0$ since $2 \le q(\alpha) + \varepsilon_i^2 q(c_i) - q(\alpha|\varepsilon_i c_i) = 2 - q(\alpha|\varepsilon_i c_i)$. Hence $2 = 2q(\alpha) = q(\alpha|\alpha) = \sum_{i:\alpha_i \neq 0} |\alpha_i| q(\alpha|\varepsilon_i c_i) \le 0$, a contradiction.

We prove now by induction on $|\alpha|$ that $\hat{N}_\alpha = N_\alpha$. For $|\alpha| = 1$, we note that $\alpha = \varepsilon c_i$ for some i, and thus $E_\alpha \in \hat{N}_\alpha$. For $|\alpha| = 0$, observe that $\pi_0(c_i) = -[E_{c_i}, E_{-c_i}] \in \hat{N}$ for any i and thus $N_0 = \hat{N}_0$.

Let now $\alpha \in R^1$ with $|\alpha| > 1$ and $\beta \in R$ be such that $|\beta| < |\alpha|$ and $\alpha - \beta = \varepsilon c_i$. By induction hypothesis, we may assume that $N_\beta = \hat{N}_\beta$. If $\beta \in R^1$, we have $\pm E_\alpha = [E_{\varepsilon c_i}, E_\beta] \in \hat{N}$ and if $\beta \in R^0$, then $\pm 2E_\alpha = [E_{\varepsilon c_i}, \pi_\beta(\varepsilon c_i)] \in \hat{N}_\alpha$. Thus, in any case $\hat{N}_\alpha = N_\alpha$.

For $\alpha \in R^0$, choose i with $\alpha_i \neq 0$ and denote by ε_i the sign of α_i. Then $\beta = \alpha - \varepsilon_i c_i \in R^1$ and by induction hypothesis $E_\beta \in \hat{N}_\beta$. Thus, we obtain that $\pm\pi_\alpha(c_i) = [E_{\varepsilon c_i}, E_\beta] \in \hat{N}_\alpha$. Now, consider any j, such that $q(c_i|c_j) \neq 0$ and calculate $[E_{c_j}, E_{-c_j}, \pi_\alpha(c_i)] = \epsilon(\alpha, c_j) q(c_i|c_j)[E_{c_j}, E_{\alpha - c_j}] = \pm q(c_i|c_j)\pi_\alpha(c_j)$ and therefore $\pi_\alpha(c_j) \in \hat{N}_\alpha$. Since q is connected, we infer inductively that $\pi_\alpha(c_j) \in \hat{N}_\alpha$ for any j and hence the result. □

Corollary 2.3. *If q is non-negative then the Lie subalgebra H of $\mathrm{G}(q)$ generated by $h_1, \ldots, h_n$ is n-dimensional, that is $H = \bigoplus_{i=1}^n \mathbb{C}h_i$.*

Proof. Since $H_1, \ldots, H_n$ are linearly independent in N, the same holds for $h_1, \ldots, h_n$. □

For a different proof of the Corollary 2.3 we refer to Slodowy's habilitation thesis [18, chapter 4].

3. Gabrielov transformations

Given a unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, we define for any $r \neq s$ and $\lambda \in \mathbb{Z}$ a linear transformation T^λ_{sr} by $T^\lambda_{sr} c_i = c_i$ for any $i \neq r$ and $T^\lambda_{sr} c_r = c_r + \lambda c_s$. Another linear transformation I_s is given by $I_s(c_i) = c_i$ for any $i \neq s$ and $I_s(c_s) = -c_s$.

Note that for $\lambda = -q_{rs}$ (where we set $q_{rs} = q_{sr}$ in case $s < r$ for the sake of simplicity), the form $q' = q \circ T^\lambda_{sr}$ is again a unit form. Following [13], we say in this case, that T^λ_{sr} is a *Gabrielov transformation* for q; such a transformation is called (weak) braid transformation in [18, chapter 4], and is sometimes called deflation if $q_{sr} = -1$ or inflation if $q_{sr} = 1$. The effect of a Gabrielov transformation on the quadratic form is exhibited by the following formulas:

$$q'_{rs} = -q_{rs}; \quad q'_{ir} = q_{ir} - q_{sr} q_{is} \text{ (for } i \neq r, s); \; q'_{ij} = q_{ij} \text{ (for } i, j \neq r).$$

Furthermore, we say that q' is obtained from q by a *sign-inversion* if $q' = q \circ I_s$ for some s. Two unit forms q and $q' : \mathbb{Z}^n \to \mathbb{Z}$ are *Gabrielov equivalent*, or *G-equivalent* for short, if there exists a sequence of unit forms $q = q^{(0)}, q^{(1)}, \ldots, q^{(t)} = q'$ such that $q^{(i)}$ is obtained from $q^{(i-1)}$ by a Gabrielov transformation, a sign-inversion or a permutation of the variables for each $i = 1, \ldots, t$. We use the notation $q \sim_{\mathrm{G}} q'$ to indicate G-equivalence.

Proof Theorem 1.1. Arguing by induction, we only have to verify the assertion for $q' = q \circ T^\lambda_{sr}$ ($\lambda = -q_{sr}$) and $q' = q \circ I_r$ (this verification is trivial if $q' = q \circ T$ and T is a permutation matrix). In both cases, we specify elements $\tilde{e}_i, \tilde{e}_{-i}, \tilde{h}_i$ ($1 \leq i \leq n$) in $\mathrm{G}(q)$ which satisfy the relations (R1)$-$(R∞) with respect to q', and hence obtain a homomorphism of graded Lie algebras $\varphi : \mathrm{G}(q') \to \mathrm{G}(q)$ mapping the generators $e'_{\varepsilon i}$ and h'_i of $\mathrm{G}(q')$ to $\tilde{e}_{\varepsilon i}$ and $\tilde{h}_i$, respectively. By a similar construction, we define $\psi : \mathrm{G}(q) \to \mathrm{G}(q')$ and show that it is the inverse of φ.

We start with the first case, $q' = q \circ T^\lambda_{sr}$, and denote by C' the Cartan matrix of q'. Moreover, we set

$$\alpha = |C_{sr}|, \quad \sigma = -\mathrm{sign}(C_{sr}), \quad \text{and hence } C_{sr} = -\sigma\alpha = -\lambda$$

and define the following elements in $\mathrm{G}(q)$

(3.1)

$$\tilde{e}_{\varepsilon i} = \begin{cases} \frac{\varepsilon^\alpha}{\alpha!} (\mathrm{ad}\, e_{\varepsilon\sigma s})^\alpha (e_{\varepsilon r}), & \text{if } i = r \\ e_{\varepsilon i}, & \text{if } i \neq r \end{cases} \quad \text{and} \quad \tilde{h}_i = \begin{cases} h_r + \sigma\alpha h_s, & \text{if } i = r \\ h_i, & \text{if } i \neq r. \end{cases}$$

Clearly, these elements satisfy the relations (R1) with respect to q'. If $i, j \neq r$ then $[\tilde{h}_i, \tilde{e}_{\varepsilon j}] = [h_i, e_{\varepsilon j}] = \varepsilon C_{ij} e_{\varepsilon j} = \varepsilon C'_{ij} \tilde{e}_{\varepsilon j}$ and $[\tilde{e}_{\varepsilon i}, \tilde{e}_{-\varepsilon i}] =$

$[e_{\varepsilon i}, e_{-\varepsilon i}] = \varepsilon h_i = \varepsilon \tilde{h}_i$, thus only cases involving the index r remain to be considered for (R2) and (R3). It follows from Lemma 2.1 for any i, that $[h_i, \tilde{e}_{\varepsilon r}] = (\alpha\sigma\varepsilon C_{is} + \varepsilon C_{ir})\tilde{e}_{\varepsilon r} = \varepsilon(C_{ir} - C_{sr}C_{is})\tilde{e}_{\varepsilon r}$. From this, we conclude for $i \neq r$, that $[\tilde{h}_i, \tilde{e}_{\varepsilon r}] = \varepsilon C'_{ir}\tilde{e}_{\varepsilon r}$ and $[\tilde{h}_r, \tilde{e}_{\varepsilon r}] = [h_r - C_{sr}h_s, \tilde{e}_{\varepsilon r}] = \varepsilon(C_{rr} - C_{sr}C_{rs} - C_{sr}(C_{sr} - C_{sr}C_{ss}))\tilde{e}_{\varepsilon r} = \varepsilon C'_{rr}\tilde{e}_{\varepsilon r}$, where we used $C_{ss} = 2$ and $C_{rs} = C_{sr}$ in the last equation. For completing (R2), we calculate $[\tilde{h}_r, \tilde{e}_{\varepsilon i}] = [h_r - C_{sr}h_s, e_{\varepsilon i}] = \varepsilon C'_{ri}e_{\varepsilon i}$.

For (R3), it remains to show that $[\tilde{e}_{\varepsilon r}, \tilde{e}_{-\varepsilon r}] = \varepsilon\tilde{h}_r$, which is the most difficult part of the proof. If we define $\zeta_k^\varepsilon = (\operatorname{ad} e_{\sigma\varepsilon s})^k(e_{\varepsilon r})$, then this follows from statement (3.5) below, in the special case where $k = \alpha$.

$$[h_s, \zeta_k^\varepsilon] = \varepsilon\sigma(2k - \alpha)\zeta_k^\varepsilon, \qquad \text{for } 0 \leq k \leq \alpha; \tag{3.2}$$

$$[e_{\varepsilon\sigma s}, \zeta_k^{-\varepsilon}] = \begin{cases} 0, & \text{for } k = 0, \\ k(\alpha - (k-1))\zeta_{k-1}^{-\varepsilon}, & \text{for } 0 < k \leq \alpha; \end{cases} \tag{3.3}$$

$$[e_{\varepsilon\sigma s}, \zeta_{k-1}^\varepsilon, \zeta_k^{-\varepsilon}] = (-1)^k\varepsilon\sigma\frac{\alpha!\,k!}{(\alpha-k)!}h_s, \qquad \text{for } 0 < k \leq \alpha; \tag{3.4}$$

$$[\zeta_k^\varepsilon, \zeta_k^{-\varepsilon}] = (-1)^k\varepsilon\frac{\alpha!\,k!}{(\alpha-k)!}(h_r + k\sigma h_s), \qquad \text{for } 0 \leq k \leq \alpha. \tag{3.5}$$

Notice that $\zeta_0^\varepsilon = e_{\varepsilon r}$. Statement (3.2) follows directly from Lemma 2.1. For the remaining statements, we use induction and only indicate the crucial Jacobi identity used in the induction step. For (3.3), use $[e_{\varepsilon\sigma s}, \zeta_k^{-\varepsilon}] = [\varepsilon\sigma h_s, \zeta_{k-1}^{-\varepsilon}] + [e_{-\varepsilon\sigma s}, e_{\varepsilon\sigma s}, \zeta_{k-1}^{-\varepsilon}]$, whereas for (3.4) use $[e_{\varepsilon\sigma s}, \zeta_{k-1}^\varepsilon, \zeta_k^{-\varepsilon}] = [e_{\varepsilon\sigma s}, e_{\varepsilon\sigma s}, \zeta_{k-2}^\varepsilon, \zeta_k^{-\varepsilon}] - [e_{\varepsilon\sigma s}, \zeta_{k-2}^\varepsilon, e_{\varepsilon\sigma s}, \zeta_k^{-\varepsilon}]$ and notice that the first summand on the right hand side equals zero, because $[\zeta_{k-2}^\varepsilon, \zeta_k^{-\varepsilon}] = 0$ by (R∞) since its degree is $-2\varepsilon\sigma c_s$. Finally, for (3.5), use $[\zeta_k^\varepsilon, \zeta_k^{-\varepsilon}] = [e_{\varepsilon\sigma s}, \zeta_{k-1}^\varepsilon, \zeta_k^{-\varepsilon}] - [\zeta_{k-1}^\varepsilon, e_{\varepsilon\sigma s}, \zeta_k^{-\varepsilon}]$.

For (R∞), we observe that, if $q(\alpha) > 1$, then $\mathrm{G}(q)_\alpha = 0$, as follows from Lemma 2.1 and (R∞) with respect to q. Let $\alpha' = \sum_{j=1}^t \varepsilon_j c_{i_j}$ be such that $q'(\alpha') > 1$. We have to show that $X = [\tilde{e}_{\varepsilon_1 i_1}, \ldots, \tilde{e}_{\varepsilon_t i_t}]$ equals zero. We have $X \in \mathrm{G}(q)_{\tilde{\alpha}}$, where $\tilde{\alpha} = \sum_{j=1}^t \varepsilon_j \tilde{c}_{i_j}$ with $\tilde{c}_a = c_a$ if $a \neq r$ and $\tilde{c}_r = c_r - C_{sr}c_s$, or shortly $\tilde{c}_a = T_{sr}^\lambda c_a$, for any a. Therefore $\tilde{\alpha} = T_{sr}^\lambda \alpha'$ and hence $q(\tilde{\alpha}) = q(T_{sr}^\lambda\alpha') = q'(\alpha') > 1$. This implies $\mathrm{G}(q)_{\tilde{\alpha}} = 0$ by the preceding remark.

We hence obtain $\varphi : \mathrm{G}(q') \to \mathrm{G}(q)$ as desired.

Notice that $q'_{sr} = -q_{sr}$, thus $\alpha = |C'_{sr}|$. Define $\tau = -\operatorname{sign}(C'_{sr})$, and observe that $\sigma\tau = -1$. Similarly as in (3.1), we define in $\mathrm{G}(q')$ the elements $\tilde{e}'_{\varepsilon r} = \frac{\varepsilon^\alpha}{\alpha!}(\operatorname{ad} e'_{\varepsilon\tau s})^\alpha(e'_{\varepsilon r})$, $\tilde{h}'_r = h'_r + \tau\alpha h'_s$ and $\tilde{e}'_{\varepsilon i} = e'_{\varepsilon i}$, $\tilde{h}'_i = h'_i$

for $i \neq r$ and obtain a homomorphism of graded Lie algebras $\psi : \mathrm{G}(q) \to \mathrm{G}(q')$, which maps $e_{\varepsilon i}$ to $\tilde{e}'_{\varepsilon i}$ and h_i to $\tilde{h}'_i$, for any i.

We finally show that $\varphi \circ \psi = \mathrm{id}$; the proof of $\psi \circ \varphi = \mathrm{id}$ is similar. For $i \neq r$, we have $\varphi \circ \psi(e_{\varepsilon i}) = e_{\varepsilon i}$ and $\varphi \circ \psi(h_i) = h_i$. Moreover, $\varphi \circ \psi(h_r) = \varphi(h'_r - C'_{sr} h'_s) = h_r - C_{sr} h_s - C'_{sr} h_s = h_r$. It follows from (3.3) by induction, that $(\operatorname{ad} e_{-\varepsilon\sigma s})^k(\zeta^\varepsilon_\alpha) = \frac{\alpha!\,k!}{(\alpha-k)!}\zeta^\varepsilon_{\alpha-k}$ for $0 \leq k \leq \alpha$. Thus $(\operatorname{ad} e_{\varepsilon\tau s})^\alpha(\zeta^\varepsilon_\alpha) = \alpha!^2 e_{\varepsilon r}$ and we have

$$\begin{aligned}
\varphi \circ \psi(e_{\varepsilon r}) =& \varphi\left(\frac{\varepsilon^\alpha}{\alpha!}(\operatorname{ad} e'_{\varepsilon\tau s})^\alpha(e'_{\varepsilon r})\right)\\
=& \frac{\varepsilon^\alpha}{\alpha!}\left(\operatorname{ad}\varphi(e'_{\varepsilon\tau s})\right)^\alpha\left(\varphi(e'_{\varepsilon r})\right)\\
=& \frac{\varepsilon^\alpha}{\alpha!}(\operatorname{ad} e_{\varepsilon\tau s})^\alpha\left(\frac{\varepsilon^\alpha}{\alpha!}(\operatorname{ad} e_{\sigma\varepsilon s})^\alpha(e_{\varepsilon r})\right)\\
=& \frac{1}{\alpha!^2}(\operatorname{ad} e_{\varepsilon\tau s})^\alpha(\zeta^\varepsilon_\alpha)\\
=& e_{\varepsilon r}.
\end{aligned}$$

It remains to deal with the case where $q' = q \circ I_r$. Observe that $q'_{ri} = -q_{ri}$ for any $i \neq r$ and $q'_{ij} = q_{ij}$ for any $i, j \neq r$. Again, the elements

$$\tilde{e}_{\varepsilon i} = \begin{cases} e_{-\varepsilon r}, & \text{if } i = r\\ e_{\varepsilon i}, & \text{if } i \neq r\end{cases} \quad\text{and}\quad \tilde{h}_i = \begin{cases} -h_r, & \text{if } i = r\\ h_i, & \text{if } i \neq r\end{cases}$$

satisfy the relations (R1)$-$(R∞) for q'. The verification is similar to the above, but substantially easier. Therefore we obtain a homomorphism of graded Lie algebras $\varphi : \mathrm{G}(q') \to \mathrm{G}(q)$ which maps $e'_{\varepsilon i}$ to $\tilde{e}_{\varepsilon i}$ and h'_i to $\tilde{h}_i$. Similarly, we define in $\mathrm{G}(q')$ the elements $\tilde{e}'_{\varepsilon r} = e_{-\varepsilon r}$, $\tilde{h}'_r = -h_r$ and $\tilde{e}'_{\varepsilon i} = e_{\varepsilon i}$, $\tilde{h}'_i = h_i$ for $i \neq r$ and obtain a homomorphism of graded Lie algebras $\psi : \mathrm{G}(q) \to \mathrm{G}(q')$ which maps $e_{\varepsilon i}$ to $\tilde{e}'_{\varepsilon i}$ and h_i to $\tilde{h}'_i$, for any i. It is straightforward to check that ψ is inverse to φ. This finishes the proof. □

Remark 3.1. In the preceding proof, in order to show that the elements $\tilde{e}_{\varepsilon i}$ and $\tilde{h}_i$ of $\mathrm{G}(q)$ satisfy the relations (R1), (R2) and (R3) with respect to q', we used the relations (R∞) of $\mathrm{G}(q)$ only to show $[\zeta^\varepsilon_{k-2}, \zeta^{-\varepsilon}_k] = 0$ for (3.4). It will be useful later (in 6.5) to observe that even the relations (R4) of $\mathrm{G}(q)$ are sufficient for deducing this.

Namely, show $[\zeta^\varepsilon_h, \zeta^{-\varepsilon}_k] = 0$ by double induction over $k \geq 2$ and $h = 0, \ldots, k-2$. For $h = 0$, using (R4) we have $[e_{\varepsilon r}, e_{-\varepsilon\sigma s}] = 0$ and hence infer inductively $[\zeta_0^\varepsilon, \zeta_k^{-\varepsilon}] = (\operatorname{ad} e_{-\varepsilon\sigma s})^i([\zeta_0^\varepsilon, \zeta_{k-i}^{-\varepsilon}])$. Thus for $i = k \geq 2$ we have $[\zeta_0^\varepsilon, \zeta_k^{-\varepsilon}] = (\operatorname{ad} e_{-\varepsilon\sigma s})^k(\varepsilon h_r) = 0$. For $h > 0$ (and hence $k > 2$), we notice that in $[\zeta_h^\varepsilon, \zeta_k^{-\varepsilon}] = [e_{\varepsilon\sigma s}, \zeta_{h-1}^\varepsilon, \zeta_k^{-\varepsilon}] - [\zeta_{h-1}^\varepsilon, e_{\varepsilon\sigma s}, \zeta_k^{-\varepsilon}]$ the second summand is a multiple of $[\zeta_{h-1}^\varepsilon, \zeta_{k-1}^{-\varepsilon}]$ by (3.3) and hence both summands are zero by induction. □

4. Equivalence of Unit Forms

In this section we focus on *non-negative* unit forms, that is, unit forms q with $q(x) \geq 0$ for all $x \in \mathbb{Z}^n$. Two unit forms $q, q' : \mathbb{Z}^n \to \mathbb{Z}$, are *equivalent*, if there exists a $\mathbb{Z}-$invertible linear transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ with $q' = q \circ T$. The *radical* of q is $\operatorname{rad} q = \{x \mid q(x+y) = q(y), \text{for all } y\}$, which is a direct summand of $\mathbb{Z}^n$.

We recall from [1] that non-negative unit forms are classified completely up to equivalence by their *corank*, that is, the rank of the radical, and their *Dynkin type*, that is a disjoint union of Dynkin diagrams $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_n$ $(n = 6, 7, 8)$. Notice that a unit form is connected if and only if its Dynkin type is connected.

Observe, that the coefficients of a non-negative unit form q are bounded: $-2 \leq q_{ij} \leq 2$. Moreover, for such forms Gabrielov transformations in double edges act like sign inversions, that is if $\lambda = -q_{sr}$ satisfies $|\lambda| = 2$ then $q \circ T_{sr}^\lambda = q \circ I_r$.

Proof of Proposition 1.2. Let q and q' be two connected non-negative unit forms and assume that they are equivalent. If their corank is zero, then $q \sim_{\mathrm{G}} Q_\Delta \sim_{\mathrm{G}} q'$, where Q_Δ is the unit form associated to a Dynkin diagram Δ, see for example [6, Theorem 6.2].

We prove by induction on the corank r that $q \sim_{\mathrm{G}} Q_\Delta[r]$, where Δ is the Dynkin type of q and $Q_\Delta[r] : \mathbb{Z}^{m+r} \to \mathbb{Z}$ is the unit form given by

$$Q_\Delta[r](v_1, \ldots, v_m, w_1, \ldots, w_r) = Q_\Delta(v_1 + \textstyle\sum_j w_j, v_2, \ldots, v_m).$$

For $r = 0$, this is already stated above. For $r > 0$, we proceed in 4 steps (i) $-$ (iv), in each showing that there exists a unit form $q' \sim_{\mathrm{G}} q$, which satisfies certain conditions.

(i) there exists a vector $v \in \operatorname{rad} q'$ with $v_n = 1$, $v_i \geq 0$ for all i.
(ii) in addition to (i), the restriction $q'^{(n)}$ of q' to the first $n-1$ variables equals $Q_\Delta[r-1]$ and $v_j = 0$ for $m < j < n$.
(iii) in addition to (ii), there exits $i \leq m$ with $q'_{in} = -2$.
(iv) $q' = Q_\Delta[r]$.

(i) Choose $v \in \operatorname{rad} q$, $v \neq 0$, such that the *support*, that is the set of vertices i with $v_i \neq 0$, is minimal. Since we may apply simultaneous sign-inversions in every vertex i for which $v_i < 0$, we can assume that $v_i \geq 0$ for every i, and further, we may assume that the entries of v are coprime. Consider now the *restriction* p of q to the support of v, by setting all other variables zero. By the minimality of the support of v, the radical of p is of rank one, $\operatorname{rad} p = \mathbb{Z}w$. It has been shown in [8], that w is a vector with $w_i = 1$ for some vertex i. Since the entries of v are coprime, we conclude that w is the restriction of v to its support, and thus $v_i = 1$. Using a permutation of the variables, we might assume $i = n$.

(ii) We assume that q itself satisfies (i). Let $p = q^{(n)}$ be the restriction of q to the first $n-1$ variables. It has been shown in [1], that p is connected again with the same Dynkin type Δ and $\operatorname{rank}(\operatorname{rad} p) = \operatorname{rank}(\operatorname{rad} q) - 1$. By induction, we assume that there is a sequence of Gabrielov transformations T taking p into $Q_\Delta[r-1]$. We may consider T also as sequence of Gabrielov transformations for q (acting on the first $n-1$ variables only), obtaining $q' = q \circ T$ with restriction $q'^{(n)} = Q_\Delta[r-1]$. If we set $v' = T^{-1}v$ then $q'(v') = q(v) = 0$, $v'_n = v_n$ and the vector $w = v' - \sum_{j=1}^{r-1} v'_{m+j}(c_{m+j} - c_1)$ satisfies $q(w) = q(v') = 0$, $w_j \geq 0$ (for $1 \leq j \leq m$), $w_j = 0$ (for $m < j < n$) and $w_n = 1$.

(iii) We assume that q itself satisfies (ii). If $v_j q_{jn} \geq 0$ for all $j \leq m$ then we would have $q(v) = v_n^2 + q^{(n)}(v - c_n) + \sum_{j=1}^{m} q_{jn} v_j v_n \geq 2$, a contradiction. Thus there exists $i \leq m$ with $v_i q_{in} < 0$. If $q_{in} = -1$ then $\tilde{q} = q \circ T_{in}^{+1}$ together with $\tilde{v} = T_{ni}^{-1} v = v - c_i$ satisfies again (ii). Since $|\tilde{v}| < |v| = \sum_j |v_j|$, after a finite number of such steps we must end up with a form $q' = q \circ T$ where $q'_{ni} = -2$ for some $i \leq m$ together with a vector satisfying (iii).

(iv) We assume that q itself satisfies (iii). Let $\tilde{q} = q \circ I_n$, obtaining $\tilde{q}_{in} = 2$. If $i = 1$ we are done: it follows from the non-negativity of $\tilde{q}$ that $\tilde{q}_{is} = \tilde{q}_{ns}$ for all $s \neq i, n$, thus $\tilde{q} = \tilde{q}^{(n)}[1] = (Q_\Delta[r-1])[1] = Q_\Delta[r]$.

If $i > 1$, we choose a shortest walk $i = i_1, \ldots, i_a = 1$ inside Δ connecting the vertex i with 1. Let $j = i_2$. By the non-negativity, we must have $\tilde{q}_{jn} = \tilde{q}_{ji} = -1$ and may thus apply T_{jn}^{+1}. The resulting unit form $\tilde{q}' = \tilde{q} \circ T_{jn}^{+1}$ satisfies $\tilde{q}'_{in} = 1$, and hence we apply T_{in}^{-1} to $\tilde{q}'$ in order to obtain $\tilde{q}'' = \tilde{q}' \circ T_{in}^{-1}$. This form satisfies $\tilde{q}''_{jn} = 2$, that is, it has a shorter walk inside Δ connecting j to 1. Since $\tilde{q}'^{(n)} = \tilde{q}^{(n)} = Q_\Delta[r-1]$, we may apply induction on the length of the walk and obtain the desired result. □

We mention, that the above result may also be derived from a result announced by Zeldich in [20].

Note that the connectivity assumption is important for the validity of the preceding result. If $q, q' : \mathbb{Z}^4 \to \mathbb{Z}$ are given by $q(x) = x_1^2 + \ldots + x_4^2 - x_1x_2 - x_2x_3 - x_1x_3$ and $q'(x) = x_1^2 + \ldots + x_4^2 - x_1x_2 - x_2x_3 - 2x_3x_4$ then q and q' are equivalent but not G-equivalent. Moreover, $\mathrm{G}(q)$ and $\mathrm{G}(q')$ cannot be isomorphic as graded Lie algebras since for any non-zero $\alpha' \in \mathbb{Z}^n$, we have $\dim \mathrm{G}(q')_{\alpha'} \leq 1$, whereas $\dim \mathrm{G}(q)_\alpha = 2$ for non-zero $\alpha \in \operatorname{rad} q$, as follows from Proposition 6.1.

Remark 4.1. Proposition 1.2 does not extend to the indefinite case: let $p(x) = x_1^2 + x_2^2 + x_3^2 - 5x_1x_2 - 2x_1x_3 - 9x_2x_3$ and $q(x) = x_1^2 + x_2^2 + x_3^2 - 5x_1x_2 - 5x_1x_3 - 5x_2x_3$. Then $p = q \circ T$, where $T(c_1) = c_1$, $T(c_2) = c_2$ and $T(c_3) = 4c_1 + 3c_2 - c_3$, hence p and q are equivalent. But, since the greatest common divisor of the non-diagonal entries of the associated Cartan matrix is preserved under Gabrielov transformations, p and q can not be G-equivalent.

5. Root space decomposition

In this section, we assume q to be non-negative. Let M be the set of monomials in $\mathrm{G}(q)$ and $H = \bigoplus_{i=1}^n \mathbb{C}h_i$, which is a commutative Lie subalgebra of $\mathrm{G}(q)$ by Proposition 2.3. For any $h = \sum_{i=1}^n \lambda_i h_i$, we define $r(h) = \sum_{i=1}^n \lambda_i c_i \in \mathbb{C}^n$. Further, we define $\langle h, \alpha \rangle = r(h)^\top C\alpha$ for any $\alpha \in \mathbb{C}^n$. By abuse of notation, we denote the obvious extension of q to $\mathbb{C}^n$ by the same symbol.

Proposition 5.1. *For all $h \in H$ and all $m \in M$, we have $[h, m] = \langle h, \deg(m) \rangle m$. Moreover, for any $\alpha \in \mathbb{Z}^n$ we have $\mathrm{G}(q)_\alpha \subseteq \{x \in \mathrm{G}(q) \mid [h, x] = \langle h, \alpha \rangle x, \forall h \in H\}$.*

Proof. Let $\widehat{\mathrm{G}}(q)$ be the free Lie algebra generated by e_i, e_{-i}, h_i $(1 \leq i \leq n)$ and let ℓ be the *length function* on $\widehat{M}$, the set of monomials in $\widehat{\mathrm{G}}(q)$, that is $\ell(e_{\varepsilon i}) = 1$, $\ell(h_i) = 1$ and inductively $\ell([x', x'']) = \ell(x') + \ell(x'')$. The canonical projection $\pi : \widehat{\mathrm{G}}(q) \to \mathrm{G}(q)$ preserves the degree and induces a surjection on the monomials.

We show by induction on $\ell(x)$ that

$$[h, \pi(x)] = \langle h, \deg(x) \rangle \pi(x)$$

for all $x \in \widehat{M}$. Let $h = \sum_{i=1}^n \lambda_i c_i$. If $\ell(x) = 1$ then $x = h_i$ (this case is clear) or $x = e_{\varepsilon j}$, and then $[h, \pi(x)] = \sum_{i=1}^n \lambda_i [h_i, e_{\varepsilon j}] = \varepsilon \sum_{i=1}^n \lambda_i C_{ij} e_{\varepsilon j} = \lambda \pi(x)$ with $\lambda = \sum_{i=1}^n \lambda_i c_i^\top C \varepsilon c_j = r(h)^\top C \deg(\pi(x)) = \langle h, \deg(x) \rangle$.

If $\ell(x) > 1$, we have $x = [x', x'']$ and set $m' = \pi(x')$, $m'' = \pi(x'')$. Therefore $[h, \pi(x)] = [m', [h, m'']] - [m'', [h, m']] = \langle h, \deg(m'') \rangle [m', m''] -$

$\langle h, \deg(m')\rangle[m'', m']$ by induction hypothesis. Hence, we have $[h, \pi(x)] = \langle h, \deg(m') + \deg(m'')\rangle[m', m''] = \langle h, \deg(\pi(x))\rangle\pi(x)$.

The remaining part follows from this and Lemma 2.1. □

In general the inclusion in Proposition 5.1 is not an equality. In order to achieve this we pass to an extension of G(q) by the $\mathbb{C}$-dual of $\operatorname{rad} q$.

Choose a projection $\rho : \mathbb{C}^n \to \operatorname{rad} q$ and set $\tilde{\mathrm{G}}^\rho(q) = \mathrm{G}(q) \oplus (\operatorname{rad} q)^*$, as a vector space, where $(\operatorname{rad} q)^*$ denotes the dual space with respect to the field $\mathbb{C}$. For $\xi, \xi' \in (\operatorname{rad} q)^*$ and $x \in \mathrm{G}(q)_\alpha$ define

$$[\xi, \xi'] = 0 \qquad \text{and} \qquad [\xi, x] = -[x, \xi] = \xi\rho(\alpha)x.$$

Lemma 5.2. *Using the Lie algebra structure on* $\mathrm{G}(q)$*, the above bracket rules induce the structure of a Lie algebra on* $\tilde{\mathrm{G}}^\rho(q)$*.*

Proof. We check the Jacobi identity. Let $\xi, \xi' \in (\operatorname{rad} q)^*$, $x \in \mathrm{G}(q)_\alpha$ and $y \in \mathrm{G}(q)_\beta$. Then, we have

$$\begin{aligned}[][\xi, \xi', x] + [\xi', x, \xi] + [x, \xi, \xi'] =& \xi'\rho(\alpha)[\xi, x] - \xi\rho(\alpha)[\xi', x] \\ =& \xi'\rho(\alpha)\xi\rho(\alpha)x - \xi\rho(\alpha)\xi'\rho(\alpha)x \\ =& 0,\end{aligned}$$

and since $[x, y] \in \mathrm{G}(q)_{\alpha+\beta}$, we obtain

$$\begin{aligned}[][\xi, x, y] + [x, y, \xi] + [y, \xi, x] =& \xi\rho(\alpha + \beta)[x, y] - \xi\rho(\beta)[x, y] + \xi\rho(\alpha)[y, x] \\ =& 0.\end{aligned}$$

This shows the statement. □

Now, set

$$\tilde{H} = H \oplus (\operatorname{rad} q)^*,$$

$\deg(\xi) = 0 \in \mathbb{Z}^n$ for any $\xi \in (\operatorname{rad} q)^*$ and define a bilinear form $\langle\_,\_\rangle : \tilde{H} \times H \to \mathbb{C}$ by $\langle h, \alpha\rangle = r(h)^\top C\alpha$ for $h \in H$ and $\langle \xi, \alpha\rangle = \xi\rho(\alpha)$ for $\xi \in (\operatorname{rad} q)^*$. Note that this form is non-degenerate in the second variable, since for $\alpha \notin \operatorname{rad} q$, we have $C\alpha \neq 0$ and thus there is an i such that $\langle h_i, \alpha\rangle \neq 0$, whereas for any non-zero $\alpha \in \operatorname{rad} q$, we have $\langle \delta_\alpha, \alpha\rangle \neq 0$, for some $\delta_\alpha \in (\operatorname{rad} q)^*$.

For a slightly different setting the proof for the next proposition is also contained in [18, chapter 4]. Here, we restrict to a context requested by the scope of the present paper.

Proposition 5.3. (i) *The algebra* $\tilde{\mathrm{G}}^\rho(q)$ *is, up to isomorphism of graded Lie algebras, independent of the choice of the projection* $\rho : \mathbb{C}^n \to \operatorname{rad} q$*, and hence denoted* $\tilde{\mathrm{G}}(q)$ *from now on.*

(ii) $\tilde{\mathrm{G}}(q)$ *admits a root space decomposition, that is, for any* $\alpha \in \mathbb{Z}^n$, *we have*

$$\tilde{\mathrm{G}}(q)_\alpha = \{x \in \tilde{\mathrm{G}}(q) \mid [h, x] = \langle h, \alpha\rangle x \text{ for all } h \in \tilde{H}\}.$$

Proof. (i) Let ρ, ρ' be two projections $\mathbb{Z}^n \to \operatorname{rad} q$ and denote by $\pi : \mathbb{C}^n \to \mathbb{C}^n/\operatorname{rad} q, \alpha \mapsto \bar{\alpha}$ the canonical projection. Note, that for any $\xi \in (\operatorname{rad} q)^*$, the composition $\xi \circ (\rho - \rho')$ factors through π, say $\xi \circ (\rho - \rho') = \chi_\xi \circ \pi$. Further observe that the linear map $\mathbb{C}^n \to \mathbb{C}^n, \alpha \mapsto C\alpha$ induces an injective map $\bar{C} : \mathbb{C}^n/\operatorname{rad} q \hookrightarrow \mathbb{C}^n$. Thus there exists a linear map $\lambda_\xi : \mathbb{C}^n \to \mathbb{C}$ with $\lambda_\xi \circ \bar{C} = \chi_\xi$, obtaining the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}^n \arrow[r, "\xi \circ (\rho - \rho')"] \arrow[d, "\pi"'] & \mathbb{C} \\
\mathbb{C}^n/\operatorname{rad} q \arrow[ur, "\chi_\xi"] \arrow[r, hook, "\bar{C}"'] & \mathbb{C}^n \arrow[u, dashed, "\lambda_\xi"']
\end{tikzcd}$$

Let $\lambda_{\xi,i} = \lambda_\xi(c_i)$ and $\eta_\xi = \sum_{i=1}^n \lambda_{\xi,i} h_i$. Choose a base $\xi_1, \ldots, \xi_r$ of $(\operatorname{rad} q)^*$ and let η_{ξ_i} be an element of H obtained in this way. Extend by linearity, that is $\eta(\xi) = \sum_{i=1}^r \mu_i \eta_{\xi_i}$ if $\xi = \sum_{i=1}^r \mu_i \xi_i$. Then, we have $\xi\rho(\alpha) = \xi\rho'(\alpha) + r\eta(\xi)^\top C\alpha$, for all $\alpha \in \mathbb{C}^n$.

Now we are ready to define $\varphi : \tilde{\mathrm{G}}^\rho(q) \to \tilde{\mathrm{G}}^{\rho'}(q)$ by $\varphi(x) = x$ for all $x \in \mathrm{G}(q)$ and $\varphi(\xi) = \xi + \eta(\xi)$ for all $\xi \in (\operatorname{rad} q)^*$. For any $x \in \mathrm{G}(q)_\alpha$, $y \in \mathrm{G}(q)_\beta$ and any $\xi, \xi' \in (\operatorname{rad} q)^*$, we have then $[\varphi(x), \varphi(y)] = [x, y] = \varphi([x, y])$, $[\varphi(\xi), \varphi(\xi')] = 0 = \varphi([\xi, \xi'])$ but also $[\varphi(\xi), \varphi(x)] = [\xi + \eta(\xi), x] = \xi\rho'(\alpha)x + r\eta(\xi)^\top C\alpha x = \xi\rho(\alpha)x = \varphi([\xi, x])$. Therefore φ is a homomorphism of graded Lie algebras. That φ is bijective is obvious.

(ii) First, we note that $[h, h'] = 0$ for any $h, h' \in \tilde{H}$. For any $x \in \tilde{\mathrm{G}}(q)_\alpha$, we have thus by definition $[h, x] = \langle h, \alpha\rangle x$ for any $h \in \tilde{H}$.

Conversely, let $x \in \tilde{\mathrm{G}}(q)$ be such that $[h, x] = \langle h, \alpha\rangle x$ for any $h \in \tilde{H}$. Write $x = \xi + \sum_{i=1}^t \lambda_i m_i$, with all $\lambda_i \neq 0$, as sum of linearly independent monomials $m_i \in \mathrm{G}(q)$ and $\xi \in (\operatorname{rad} q)^*$. Thus, we have on one hand $[h, x] = \sum_{i=1}^t \lambda_i \langle h, \deg(m_i)\rangle m_i$ by Lemma 5.1. And on the other hand, we have $[h, x] = \langle h, \alpha\rangle x = \langle h, \alpha\rangle \xi + \sum_{i=1}^t \lambda_i \langle h, \alpha\rangle m_i$. Comparing the coefficients, we obtain $\langle h, \alpha\rangle = \langle h, \deg(m_i)\rangle$ for any $h \in \tilde{H}$, and thus $\deg(m_i) = \alpha$ for any i. Therefore, if $\xi = 0$ we have $x \in \mathrm{G}(q)_\alpha \subseteq \tilde{\mathrm{G}}(q)_\alpha$ and in case $\xi \neq 0$ we have $\langle h, \alpha\rangle = 0$ for all $h \in \tilde{H}$ and hence $\alpha = 0$, which implies $x \in \tilde{\mathrm{G}}(q)_0$. $\square$

6. Non-negative forms of small corank

It is well-known, that the (simply-laced) finite-dimensional simple Lie algebras $L(\Delta)$ of Dynkin type $\Delta = \mathbb{A}_n$ $(n \geq 1)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_n$ $(n = 6, 7, 8)$ are described by generators and Serre relations, see for example [17]. Similarly, the affine Kac-Moody Lie algebras are described by generators and Serre relations. The simply-laced cases are classified by extended Dynkin diagrams $\tilde{\Delta}$, where Δ is as above, and will be denoted by $L(\tilde{\Delta})$, see [9, 12].

Proposition 6.1. *For any Dynkin diagram Δ, we have $L(\Delta) = \mathrm{G}(Q_\Delta)$ and $L(\tilde{\Delta}) = \tilde{\mathrm{G}}(Q_{\tilde{\Delta}})$.*

Proof. The generators of $L(\Delta)$ (resp. $L(\tilde{\Delta})$) satisfy the Serre relations (R1)$-$(R4) and because of the well-known root space decomposition in $L(\Delta)$ (resp. $L(\tilde{\Delta})$) all relations of (R∞) are fulfilled, thus we get the result. $\square$

As a third illustration we will show that also the (simply-laced) elliptic Lie algebras $\tilde{\mathfrak{e}}(\Gamma(R, G))$, described by Saito and Yoshii in [15], are in fact of the form $\tilde{\mathrm{G}}(q)$. For that sake, we will keep the notation from [15], except for multibrackets, which we use in reverse order. Let $\Delta(R)$ be the Dynkin diagram such that the corresponding extended Dynkin diagram $\tilde{\Delta}$ is Γ_{af}.

Before entering the proof, we will show a helpful result. Let M be the set of monomials in $\mathrm{G}(q)$ and $\overline{?} : M \to M$ the linear map defined by $\overline{e_{\varepsilon i}} = e_{-\varepsilon i}$, $\overline{h_i} = h_i$ for monomials of length one and $\overline{[m, n]} = -[\overline{m}, \overline{n}]$ inductively for monomials of greater length (the function is first defined for the free Lie algbera $\widehat{\mathrm{G}}(q)$ generated by e_i, e_{-i}, h_i (for $1 \leq i \leq n$) but passes to the quotient $\mathrm{G}(q)$ since $\overline{m} \in I$ for any $m \in I$, the ideal generated by the relations (R1)$-$(R∞)). Notice that $\deg(\overline{m}) = -\deg(m)$ for all monomials m. Let $M^1_{\varepsilon i} = \{e_{\varepsilon i}\}$ and inductively for $\alpha \in R^1$, $M^1_\alpha = \{[m, n] \mid \alpha', \alpha'' \in R^1, \alpha = \alpha' + \alpha''$ and $m \in M^1_{\alpha'}$, $n \in M^1_{\alpha''}\}$. Finally, let $M^1 = \bigcup_{\alpha \in R^1} M^1_\alpha$.

Lemma 6.2. *Let q be a non-negative unit form. For any $\alpha = \sum_{i=1}^n \alpha_i c_i$ and any $m \in M^1_\alpha$, we have $[m, \overline{m}] = \sum_{i=1}^n \alpha_i h_i =: h(\alpha)$.*

Proof. Let $x \in \widehat{\mathrm{G}}(q)$ be a monomial with $\pi(x) = m \in M^1_\alpha$ (as earlier, $\pi : \widehat{\mathrm{G}}(q) \to \mathrm{G}(q)$ denotes the canonical projection). The proof is done by induction over the length $\ell(x)$ of x.

If $\ell(x) = 1$ then $x = e_{\varepsilon i} = m$ and the statement is just (R3). For $\ell(x) > 1$, write $x = [x', x'']$ with $\deg(x') = \alpha'$, $\deg(x'') = \alpha''$, $\alpha = \alpha' + \alpha''$ and set $m' = \pi(x')$ and $m'' = \pi(x'')$.

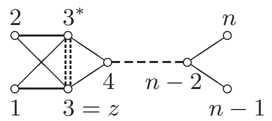
Since $1 = q(\alpha) = q(\alpha') + q(\alpha'') + q(\alpha'|\alpha'')$, we infer that $q(\alpha'|\alpha'') = -1$. Therefore we have $q(\alpha' - \alpha'') = 3$ and hence $[m', \overline{m''}] = 0$, $[m'', \overline{m'}] = 0$ by (R∞). Using induction hypothesis in the forth equation, Proposition 5.1 in the fifth and ${\alpha'}^{\top} C \alpha'' = q(\alpha'|\alpha'') = -1$ in the seventh, we obtain

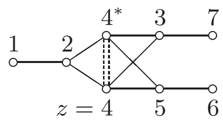
$$\begin{aligned}
[m, \overline{m}] =& - [[m', m''], [\overline{m'}, \overline{m''}]] \\
=& [m'', m', \overline{m'}, \overline{m''}] - [m', m'', \overline{m'}, \overline{m''}] \\
=& - [m'', \overline{m''}, m', \overline{m'}] - [m', \overline{m'}, m'', \overline{m''}] \\
=& - [m'', \overline{m''}, h(\alpha')] - [m', \overline{m'}, h(\alpha'')] \\
=& \langle h(\alpha'), -\alpha'' \rangle [m'', \overline{m''}] + \langle h(\alpha''), -\alpha' \rangle [m', \overline{m'}] \\
=& - {\alpha'}^{\top} C \alpha'' h(\alpha'') - {\alpha''}^{\top} C \alpha' h(\alpha') \\
=& h(\alpha') + h(\alpha'') \\
=& h(\alpha)
\end{aligned}$$

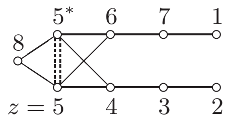
This finishes the proof. □

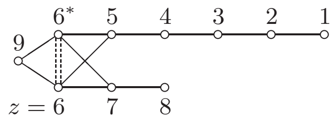
Proposition 6.3. *The Lie algebra* $\tilde{\mathfrak{e}}(\Gamma(R, G))$*, described in* [15]*, with* $\Delta(R) = \mathbb{D}_n$ *(*$n \geq 4$*) or* $\Delta(R) = \mathbb{E}_n$ *(*$n = 6, 7, 8$*), is isomorphic to* $\tilde{\mathrm{G}}(q)$*, where* q *is non-negative of corank 2 and Dynkin type* $\Delta(R)$*.*

Proof. To the different cases of $\Gamma_{\mathrm{af}} = \tilde{\Delta}$, we associate a unit form $q : \mathbb{Z}^{n+1} \to \mathbb{Z}$ given by its bigraph, see [1]:

$\tilde{\mathbb{D}}_n$:


$\tilde{\mathbb{E}}_6$:


$\tilde{\mathbb{E}}_7$:


$\tilde{\mathbb{E}}_8$:


In any case we have that q is non-negative of corank 2 and Dynkin type Δ.

In order to define a homomorphism of Lie algebras $\varphi : \tilde{\mathfrak{e}}(\Gamma(R, G)) \to \tilde{\mathrm{G}}(q)$, we define $\varphi(E^{\pm \alpha_{i-1}}) = e_{\pm i}$ $(1 \leq i \leq n)$, $\varphi(E^{\pm \alpha^*_{z-1}}) = e_{\pm z^*}$, and $\varphi(h) = h$, the latter being an abuse of notation, since we identified $\tilde{\mathfrak{h}}$ with $\tilde{H}$.

In case $\mathbb{D}_n$ with $n > 4$, we define inductively for $j = 4, \dots, n-2$

$$\varphi(E^{\alpha^*_{\varepsilon j}}) = e_{\varepsilon j^*} := [e_{\varepsilon j}, [e_{\varepsilon (j-1)^*}, e_{-\varepsilon (j-1)}]]$$

First notice that $\deg(e_{j^*}) = c_j + c_{z^*} - c_z$ for any $j > z$. Thus, by Lemma 6.2, we have $[e_{j^*}, e_{-j^*}] = h_j + h_{z^*} - h_z =: h_{j^*}$ for all $j > z$, since $e_{j^*} = [e_{-\varepsilon(j-1)}, e_{\varepsilon(j-1)^*}, e_{\varepsilon j}] \in M^1$.

We have to verify that the images of E^{α_i} satisfy the relations [15, (4.1.1)]. The relations **0.** are clear. For **I.**, we see that $\langle h_{j^*}, \alpha\rangle = \langle h_j, \alpha\rangle$ for all j.

That the first equation of **II.1.** is satisfied, follows from the definition of h_{j^*}. The second equation of **II.1**, as well as **II.2.**, **III.**, **IV.** follow from the fact that all the occurring multibracket expressions describe monomials of degree α with $q(\alpha) > 1$. Finally, for **V.**, we see that the condition is empty unless we are in case $\Delta = \mathbb{D}_n$ for $n > 4$. Then we only have to verify the case where $\alpha = j$ and $\beta = j-1$, since the other one is just given by the definition of $e_{\varepsilon j^*}$. Indeed, using $c_{z^*} - c_z \in \operatorname{rad} q$ we have $[e_{\varepsilon j}, e_{\varepsilon j^*}] = 0$, $[e_{-\varepsilon j}, e_{\varepsilon (j-1)^*}] = 0$ and $[[e_{\varepsilon (j-1)}, e_{\varepsilon j}], e_{\varepsilon j^*}, e_j] = 0$ because of (R∞), and since q is non-negative with $C_{z^* z} = 2\ C_{z^* i} = C_{zi}$ holds for all i and therefore $[h_i, e_{\varepsilon j^*}] = [h_i, e_{\varepsilon j}]$. Using our multibracket notation, we have

$$\begin{aligned}
[e_{\varepsilon(j-1)}, [e_{\varepsilon j^*}, e_{-\varepsilon j}]] =& - [e_{\varepsilon(j-1)}, e_{-\varepsilon j}, e_{\varepsilon j}, e_{\varepsilon(j-1)^*}, e_{-\varepsilon(j-1)}] \\
=& - [e_{-\varepsilon j}, e_{\varepsilon(j-1)}, e_{\varepsilon j}, e_{\varepsilon(j-1)^*}, e_{-\varepsilon(j-1)}] \\
=& - [e_{-\varepsilon j}, e_{\varepsilon j}, e_{\varepsilon(j-1)}, e_{\varepsilon(j-1)^*}, e_{-\varepsilon(j-1)}] \\
& - [e_{-\varepsilon j}, [e_{\varepsilon(j-1)}, e_{\varepsilon j}], e_{\varepsilon(j-1)^*}, e_{-\varepsilon(j-1)}] \\
=& - [e_{-\varepsilon j}, e_{\varepsilon j}, e_{\varepsilon(j-1)^*}, \varepsilon h_{j-1}] \\
& - [e_{-\varepsilon j}, e_{\varepsilon(j-1)^*}, [e_{\varepsilon(j-1)}, e_{\varepsilon j}], e_{-\varepsilon(j-1)}] \\
=& - [e_{-\varepsilon j}, e_{\varepsilon j}, -2e_{\varepsilon(j-1)^*}] \\
& + [e_{-\varepsilon j}, e_{\varepsilon(j-1)^*}, e_{\varepsilon j}, \varepsilon h_{j-1}] \\
=& - 2[e_{-\varepsilon j}, e_{\varepsilon(j-1)^*}, e_{\varepsilon j}] + [e_{-\varepsilon j}, e_{\varepsilon(j-1)^*}, e_{\varepsilon j}] \\
=& - [e_{\varepsilon(j-1)^*}, e_{-\varepsilon j}, e_{\varepsilon j}] \\
=& \varepsilon[e_{\varepsilon(j-1)^*}, h_j] \\
=& e_{\varepsilon(j-1)^*}
\end{aligned}$$

This calculation shows, that φ is indeed a homomorphism. It is clearly surjective. Conversely, we may define $\psi : \tilde{\mathrm{G}}(q) \to \tilde{\mathfrak{e}}(\Gamma(R, G))$ by $\psi(e_{\varepsilon i}) = E^{\varepsilon\alpha_{i-1}}$ for any vertex i of q and $\psi(h) = h$. In order to see that ψ is a homomorphism, we have to verify that relations (R1)–(R∞)

are satisfied. For (R1), (R2) and (R3), this is clear, whereas for (R∞) it follows from the root space decomposition of $\tilde{\mathfrak{e}}(\Gamma(R,G))$, proved in [15]: namely, that there is no root space for α with $q(\alpha) > 1$. That all $E^{\alpha_j^*}$ lie in the image of ψ follows from [15, (4.1.1) **V.**]. Thus ψ is also surjective. Clearly, ϕ and ψ are inverse to each other. □

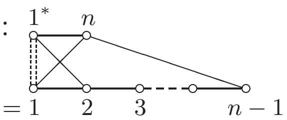
Proof of Theorem 1.3. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a non-negative connected unit form of corank $r \leq 2$ and Dynkin type Δ. Further let $q' = Q_\Delta$ in case $r = 0$, $q' = Q_{\tilde{\Delta}}$ if $r = 1$, or let q' be the form defined in the proof of Proposition 6.3 in case $r = 2$ and $\Delta = \mathbb{D}_n$ ($n \geq 4$) or $\Delta = \mathbb{E}_n$ ($n = 6, 7, 8$). It follows from [1] that q is equivalent to q', hence by Proposition 1.2 they are G-equivalent and so by Theorem 1.1, $\tilde{\mathrm{G}}(q)$ and $\tilde{\mathrm{G}}(q')$ are isomorphic as graded Lie algebras. The assertions (a), (b) and (c)(i) follow thus from the Propositions 6.1 and 6.3. In the remaining case where $r = 2$ and $\Delta = \mathbb{A}_n$ ($n \geq 2$) the result follows from Proposition 2.2 and the fact shown in [15] that $\tilde{\mathfrak{e}}(\Gamma(R,G))$ is isomorphic to the Borcherds algebra N. □

Remark 6.4. We shall now discuss briefly why we get a weaker result in the case (c)(ii), where $\Delta = \mathbb{A}_n$ ($n \geq 2$).

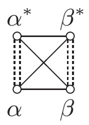
There is a problem with the relations of type **V.** which are needed to ensure the existence of a homomorphism $\varphi : \tilde{\mathfrak{e}}(\Gamma(R,G)) \to \tilde{\mathrm{G}}(q)$ in the proof of Proposition 6.3.

To be more precise, the appropiate quadratic form q is given by the following bigraph:

$\tilde{\mathbb{A}}_n$:



Proceeeding in complete analogy with the proof of Proposition 6.3, defining the elements $e_{\varepsilon j^*} := [e_{\varepsilon j}, [e_{\varepsilon(j-1)^*}, e_{-\varepsilon(j-1)}]]$ for $j = 2, \dots, n$ one has to check the validity of the relations of type **V.**



for $\alpha = \beta - 1$ ($\alpha = 1, \dots, n-1$) and for $\alpha = \beta + 1$ ($\alpha = 2, \dots, n$). In case $\{\alpha, \beta\} = \{1, n\}$ this could not be done. □

The fact that the relations are described by an infinite set (R∞) is unsatisfactory. In the discussed cases (a), (b) or (c)(i) of Theorem 1.3, q is G-equivalent to a form p where a finite set of relations is sufficient.

As we will show, this implies that also for q a finite subset of relations (R1)$-$(R∞) suffices.

More precisely, denote by $(\mathrm{R}u)_q$ ($u = 1, 2, \ldots, \infty$) the set of relations defined by the quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$, by $\widehat{\mathrm{G}}(q)$ the free Lie algebra generated by $\{e_i, e_{-i}, h_i \mid 1 \le i \le n\}$ and by $I(q)$ the ideal of $\widehat{\mathrm{G}}(q)$ generated by $(\mathrm{R}1)_q$$-$$(\mathrm{R}\infty)_q$.

Lemma 6.5. *If T is a Gabrielov transformation, a sign-inversion or a permutation for q and $q' = q \circ T$, then the ideal $I(q)$ is generated by a finite subset of relations* $(\mathrm{R}1)_q$$-$$(\mathrm{R}\infty)_q$ *if and only if $I(q')$ is generated by a finite subset of relations* $(\mathrm{R}1)_{q'}$$-$$(\mathrm{R}\infty)_{q'}$.

Proof. For $u = 3, \ldots, \infty$, let $I_u(q)$ be the ideal of $\widehat{\mathrm{G}}(q)$ generated by $(\mathrm{R}1)_q$$-$$(\mathrm{R}u)_q$ and $\mathrm{G}_u(q) = \widehat{\mathrm{G}}(q)/I_u(q)$ be the quotient Lie algebra and set $I_0(q) = 0$, $\mathrm{G}_0(q) = \widehat{\mathrm{G}}(q)$ We further denote $\pi_{uv} : \mathrm{G}_v(q) \to \mathrm{G}_u(q)$ the canonical projection for $v \le u$. Similarly, we define $I_u(q')$, $\mathrm{G}_u(q')$ and π'_{uv}.

We first deal with the case, where $T = T^{\sigma}_{rs}$ is a Gabrielov transformation for q. Define $\tilde{e}_{\varepsilon i}$ and $\tilde{h}_i$ as in (3.1) in the proof of Theorem 1.1. We have then a morphism $\Phi_{00} : \widehat{\mathrm{G}}(q') \to \widehat{\mathrm{G}}(q)$, which maps $e'_{\varepsilon i}$ to $\tilde{e}_{\varepsilon i}$ and h'_i to $\tilde{h}_i$.

The proof of the first part of Theorem 1.1 together with Remark 3.1, show that the elements $\tilde{e}_{\varepsilon i}$ and $\tilde{h}_i$ in $G_4(q)$ satisfy the relations $(\mathrm{R}1)_{q'}$$-$$(\mathrm{R}3)_{q'}$, that is Φ_{00} induces a surjective morphism $\Phi_{43} : \mathrm{G}_3(q') \to \mathrm{G}_4(q)$ making the following diagram on the left hand side commutative, where φ is the isomorphism from the proof of Theorem 1.1:

$$\begin{tikzcd}
\widehat{\mathrm{G}}(q') \arrow[rr, "\Phi_{00}"] \arrow[d, "\pi'_{30}"'] & & \widehat{\mathrm{G}}(q) \arrow[dd, "\pi_{40}"] \\
\mathrm{G}_3(q') \arrow[drr, "\Phi_{43}"] \arrow[dd, "\pi'_{\infty 3}"'] & & \\
 & & \mathrm{G}_4(q) \arrow[d, "\pi_{\infty 4}"] \\
\mathrm{G}(q') \arrow[rr, "\varphi"'] & & \mathrm{G}(q)
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{\mathrm{G}}(q') \arrow[rr, "\Phi_{00}"] \arrow[d, "\pi'_{30}"'] & & \widehat{\mathrm{G}}(q) \arrow[dd, "\pi_{40}"] \\
\mathrm{G}_3(q') \arrow[drr, "\Phi_{43}"] \arrow[ddd, "\pi'_{\infty 3}"'] & & \\
 & & \mathrm{G}_4(q) \arrow[d, "\pi_{54}"] \\
 & & \mathrm{G}_5(q) \arrow[d, "\pi_{\infty 5}"] \\
\mathrm{G}(q') \arrow[rr, "\varphi"'] \arrow[urr, "\Phi_{5\infty}"] & & \mathrm{G}(q)
\end{tikzcd}$$

Suppose now that there exists a finite subset of relations of $(\mathrm{R}1)_{q'}$$-$$(\mathrm{R}\infty)_{q'}$ generating $I(q')$, that is $I(q') = \langle (\mathrm{R}1)_{q'}, (\mathrm{R}2)_{q'}, (\mathrm{R}3)_{q'}, \rho'_1, \ldots, \rho'_N \rangle$, where each ρ'_i belongs to $(\mathrm{R}\infty)_{q'}$.

Denote by t'_j the length and by α'_j the degree of ρ'_j. Further set $\rho_j = \Phi_{00}(\rho'_j)$. Notice that the length t_j and degree α_j of ρ_j satisfy $t_j \leq (|q_{sr}| + 1)t'_j$ and $T\alpha_j = \alpha'_j$, since $\deg(\Phi_{00}(x)) = T(\deg(x))$ for any monomial x. Therefore, $q(\alpha_j) = q(\deg(\rho_j)) = q(T(\deg(\rho'_j))) = q'(\alpha'_j) > 1$ and we have that $\{\rho_1, \ldots, \rho_N\} \subset (\mathrm{R5})_q \subset (\mathrm{R}\infty)_q$, where $(\mathrm{R5})_q$ is the following set of relations

$(\mathrm{R5})_q$ $[e_{\varepsilon_1 i_1}, \ldots e_{\varepsilon_t i_t}] = 0$ if for some $j = 1, \ldots, N$, we have $\deg([e_{\varepsilon_1 i_1}, \ldots e_{\varepsilon_t i_t}]) = \alpha_j$ and $t \leq t_j$.

Then, by construction, the elements $\tilde{e}_{\varepsilon i}$ and $\tilde{h}_i$ in $G_5(q)$ satisfy the relations $(\mathrm{R1})_{q'}$–$(\mathrm{R3})_{q'}, \rho'_1, \ldots, \rho'_N$ and therefore, we obtain a morphism $\Phi_{5\infty} : \mathrm{G}(q') \to \mathrm{G}_5(q)$ making the diagram above on the right hand side commutative.

Since $\varphi = \pi_{\infty 5}\Phi_{5\infty}$ is an isomorphism, we have that $\Phi_{5\infty}$ is injective whereas the surjectivity follows from the fact that Φ_{43} and π_{54} are both surjective. Hence $\Phi_{5\infty}$ is an isomorphism, which shows that $\pi_{\infty 5}$ is an isomorphism. Since $(\mathrm{R1})_q$–$(\mathrm{R5})_q$ are all finite, we see that $I(q) = I_5(q)$ is generated by a finite subset of relations $(\mathrm{R1})_q$–$(\mathrm{R}\infty)_q$.

The case, where T is a sign-inversion or a permutation is straightforward and thus the result follows. □

Proposition 6.6. *If q is a non-negative, connected unit form of corank less or equal than 2, where for corank 2 it is assumed that the Dynkin type is not $\mathbb{A}_n$, then there exists a finite subset of relations of* (R1)–(R∞) *which is sufficient to define* $\mathrm{G}(q)$*.*

Proof. Follows directly from Proposition 1.2, Theorem 1.1, the preceding lemma and the Propositions 6.1, 6.3. □

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