GRADED FACTORIAL ALGEBRAS OF DIMENSION TWO

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Abstract. Extending a theorem of S. Mori [6] we characterize two-dimensional factorial algebras which are graded by a finitely generated abelian group of rank one and which are affine over an algebraically closed field. The result is related to the classification of weighted projective lines [2, 5] and to the topic of canonical algebras [7].

1. Introduction

Let k be an algebraically closed field. Let H be a finitely generated abelian group of rank one. For an H-graded algebra R over k we consider the following conditions:

(F1) R is a commutative graded factorial domain, that is, R is graded integral and each nonzero homogeneous element in R is a product of homogeneous primes.

(F2) R has graded Krull dimension two.

(F3) R is affine over k such that $R_0 = k$ and each homogeneous unit in R has degree zero. Moreover, the group H is generated by the elements $h \in H$ with $R_h \neq 0$.

Note that a $\mathbb{Z}$-graded algebra is graded factorial if and only if it is factorial in the ungraded sense, a fact no longer true in case the grading group will have torsion. Following [2] we introduce for a sequence of positive integers $\mathbf{p} = (p_1, \ldots, p_t)$ the abelian group $\mathbb{L}(\mathbf{p})$ which is given by generators $\vec{x}_1, \ldots, \vec{x}_t$ and relations $p_1\vec{x}_1 = \cdots = p_t\vec{x}_t$. We consider the graded algebras $\mathbb{S}(\mathbf{p}, \underline{\lambda}) = k[X_1, \ldots, X_t]/(f_3, \ldots, f_t)$, where

$$f_i = X_i^{p_i} - X_2^{p_2} + \lambda_i X_1^{p_1} \quad (i = 3, \ldots, t),$$

and $\underline{\lambda} = (\lambda_3, \ldots, \lambda_t)$ is a sequence of pairwise different nonzero scalars. $\mathbb{S}(\mathbf{p}, \underline{\lambda})$ is $\mathbb{L}(\mathbf{p})$-graded by $\deg X_i = \vec{x}_i$. It is shown in [2] that the algebras $\mathbb{S}(\mathbf{p}, \underline{\lambda})$ satisfy conditions (F1)–(F3).

Theorem 1. *Let R be an H-graded k-algebra satisfying conditions (F1)–(F3). Then there is a weight sequence $\mathbf{p} = (p_1, \ldots, p_t)$ and a parameter sequence $\underline{\lambda} = (\lambda_3, \ldots, \lambda_t)$ of pairwise different nonzero scalars, such that H and $\mathbb{L}(\mathbf{p})$ are isomorphic as groups, and such that R and $\mathbb{S}(\mathbf{p}, \underline{\lambda})$ are isomorphic as graded algebras.*

Note that $\mathbb{L}(\mathbf{p}) \simeq \mathbb{Z}$ if and only if the p_i are pairwise coprime. Theorem 1 therefore has the following consequence due to S. Mori [6].

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Corollary 2 (Mori [6]). *Let R be a positively $\mathbb{Z}$-graded algebra which is finitely generated over k with $R_0 = k$. Moreover assume that $\{n \in \mathbb{N} \mid R_n \neq 0\}$ generates $\mathbb{Z}$. Assume that R is a graded factorial domain of dimension two. Let r be the minimum of numbers of homogeneous elements which generate R. Then*
(1) *If $r = 2$, then $R \simeq k[X, Y]$, where $(\deg X, \deg Y) = 1$.*
(2) *If $r \geq 3$, then*

$$R \simeq k[X_1, \ldots, X_r]/(X_1^{p_1} + \lambda_3 X_2^{p_2} + X_3^{p_3}, \ldots, X_1^{p_1} + \lambda_r X_2^{p_2} + X_r^{p_r}),$$

where $p_1, \ldots, p_r$ are pairwise coprime and $\deg X_i = (\prod_{j=1}^r p_j)/p_i$ for $i = 1, \ldots, r$ and $\lambda_3 = 1$, $\lambda_4, \ldots, \lambda_r \in k^*$ are pairwise different.

The $\mathbb{Z}$-graded factorial algebra $\mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2)$ over the real numbers, where the generators have degree one (see [1, Ch. VII, §3, Ex. 6]), shows that the theorem and the corollary will not extend to the case where k is not algebraically closed.
We prove Theorem 1 by exploiting the graded sheaf theory over the graded projective spectrum $\mathbb{X} = \mathrm{Proj}^H(R)$. The core of the argument is to show that R is isomorphic to the polynomial algebra in two indeterminates graded by total degree in the so-called homogeneous case, which is the case where all simple sheaves have self extensions.
The homogeneous prime elements of $\mathbb{S}(\mathbf{p}, \underline{\lambda})$ are explicitly given in [2]. In the same paper the relation to the subject of canonical algebras is explained. The question when different weight sequences and different parameter sequences lead to isomorphic graded algebras is treated in [3, Prop. 9.1.].
Recently, the category of coherent sheaves over the weighted projective line $\mathbb{X}(\mathbf{p}, \underline{\lambda})$ (see [2]) has been characterized by Lenzing [5]. This work provided an additional motivation for the present paper.

2. Graded Sheaf Theory

Assume that R satisfies (F1)–(F3). Let $R = k[\pi_1, \ldots, \pi_n]$ with $\pi_1, \ldots, \pi_n$ homogeneous. Without loss of generality we can assume that also the following properties are satisfied.

(F4) The elements $\pi_1, \ldots, \pi_n$ are prime.
(F5) The elements $\pi_1, \ldots, \pi_n$ are pairwise non-associated.

Moreover, H carries a natural structure of an ordered group by setting $g \leq h$ if and only if $R_{h-g} \neq 0$. This is the case if and only if there are $\alpha_1, \ldots, \alpha_n \geq 0$ such that $h - g = \sum_{i=1}^n \alpha_i \vec{x}_i$, where we denote by $\vec{x}_i$ the degree of π_i ($i = 1, \ldots, n$).
Let tH be the torsion subgroup of H. Since $H/tH \simeq \mathbb{Z}$ and the ordering on H induces an ordering on $\mathbb{Z}$ it follows that each component R_h ($h \in H$) is of finite dimension.
Let $\mathbb{X} = \mathrm{Proj}^H(R)$ be the projective spectrum of R, that is, the set of all homogeneous prime ideals which are different from $\mathrm{rad}^H R = (\pi_1, \ldots, \pi_n)$ (compare [3, §7]). Since R has graded Krull dimension two, $\mathbb{X}$ has dimension one. Since R is graded factorial, each homogeneous prime ideal of height 1 is principal and $\mathbb{X}$ is non-singular (that is, R_x is graded regular for all $x \in \mathbb{X}$).

In the following let $x_i \in \mathbb{X}$ be the closed point which corresponds to π_i for each $i = 1, \ldots, n$. If we write $x \in \mathbb{X}$ we tacitly assume that x is a closed point, that is, x is not generic. For $x \in \mathbb{X}$ we denote by π_x a corresponding homogeneous prime element in R which is unique up to scalars. For each $x \in \mathbb{X}$ we denote by $\vec{x}$ the degree of π_x in R. The abelian group H is generated by the degrees $\vec{x}_i$ of π_i $(i = 1, \ldots, n)$. Similar, for a homogeneous element $r \in R \setminus \{0\}$ we denote by $\vec{r}$ the degree of r.

Denote by $\operatorname{coh}^H(\mathbb{X})$ the abelian category of H-graded coherent sheaves on $\mathbb{X}$. The group H acts on $\operatorname{coh}^H(\mathbb{X})$ as a group of automorphisms by shifting degrees; the shift of $\mathcal{F} \in \operatorname{coh}^H(\mathbb{X})$ by $h \in H$ is denoted by $\mathcal{F}(h)$. We have $\Gamma_H(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(h)) \simeq R(h)$ for each $h \in H$ as in [2, equation (1.5.5)], where Γ_H is defined as in [3, §7]. We remark that most of the statements proved in [2] remain valid in our setting, compare also [3].

The next lemma follows by graded factoriality. Denote by $\operatorname{Pic}\mathbb{X}$ the Picard group of $\mathbb{X}$, that is, the set of isoclasses of line bundles in $\operatorname{coh}^H(\mathbb{X})$.

Lemma 3. *The assignment $h \mapsto \mathcal{O}(h)$ induces an isomorphism $H \longrightarrow \operatorname{Pic}\mathbb{X}$.*

Proposition 4. *Let $\pi \in R \setminus \{0\}$ homogeneous. Then*
(1) *Multiplication by π induces a monomorphism $\mathcal{O} \longrightarrow \mathcal{O}(\vec{\pi})$.*
(2) *The cokernel is simple if and only if π is a homogeneous prime element; the cokernel then is concentrated in $R\pi \in \mathbb{X}$.*
(3) *Each simple sheaf is up to shift such a cokernel.*
(4) *Let $\mathcal{S}$ be a simple sheaf which is concentrated in $x \in \mathbb{X}$. If $y \in \mathbb{X}$ and $y \neq x$ then multiplication by π_y induces an isomorphism $\mathcal{S} \simeq \mathcal{S}(\vec{y})$.*

Proof. (1) Since R is graded integral, it is clear that multiplication by π induces a monomorphism $\mathcal{O} \longrightarrow \mathcal{O}(\vec{\pi})$.

(2), (4) Consider the exact sequence $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(\vec{\pi}) \longrightarrow \mathcal{S} \longrightarrow 0$. If π is prime, it induces for $x \in \mathbb{X}$ an isomorphism $R_x \longrightarrow R_x(\vec{\pi})$, which means $\mathcal{S}_x = 0$, if and only if $x \neq R\pi$; if $x = R\pi$, then πR_x is maximal in $R_x(\vec{\pi})$, hence $\mathcal{S}_x$ simple. Conversely, let $\mathcal{S}$ be simple. Assume $\pi = r \cdot s$ with r, $s \in R$ homogeneous. We then have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r, "\cdot\pi"] \arrow[d, "\cdot s"'] & \mathcal{O}(\vec{\pi}) \arrow[r] \arrow[d, equal] & \mathcal{S} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(\vec{s}) \arrow[r, "\cdot r"] & \mathcal{O}(\vec{\pi}) \arrow[r] & \mathcal{S}' \arrow[r] & 0.
\end{tikzcd}$$

It follows that $\mathcal{S}' = 0$ or $\mathcal{S}' \simeq \mathcal{S}$, whence r or s is a unit.

(3) Let $\mathcal{S}$ be a simple sheaf. There is a line bundle $\mathcal{L}_1$ such that $\operatorname{Hom}_{\mathbb{X}}(\mathcal{L}_1, \mathcal{S}) \neq 0$ (compare [2, Cor. 1.8.3] and Serre's theorem [3, Thm. 7.3]), hence there is a second line bundle $\mathcal{L}_2$ with an exact sequence $0 \longrightarrow \mathcal{L}_2 \xrightarrow{f} \mathcal{L}_1 \longrightarrow \mathcal{S} \longrightarrow 0$. Then by Lemma 3 there are h_1, $h_2 \in H$ with $\mathcal{L}_1 \simeq \mathcal{O}(h_1)$, $\mathcal{L}_2 \simeq \mathcal{O}(h_2)$. Since $\operatorname{Hom}_{\mathbb{X}}(\mathcal{O}(h_2), \mathcal{O}(h_1)) = R_{h_1 - h_2}$, the morphism f is induced by multiplication with a homogeneous element. Then apply (2). □

Proposition 5. *Let $x \in \mathbb{X}$. There are (up to isomorphism) only finitely many simple sheaves which are concentrated in x. If $x \notin \{x_1, \dots, x_n\}$, then there is (up to isomorphism) exactly one simple sheaf concentrated in x.*

We denote by $p(x)$ the number of isoclasses of simple sheaves which are concentrated in x and call $p : \mathbb{X} \longrightarrow \mathbb{N}$ the weight function of $\mathbb{X}$.

Proof. Let x, $y \in \mathbb{X}$ with $y \neq x$. Then $\vec{y} > 0$, and by the assumption on H we have that $H/\mathbb{Z}\vec{y}$ is finite. Let $\mathcal{S}$ be a simple sheaf which is concentrated in x and $H_x = \{h \in H \mid \mathcal{S} \simeq \mathcal{S}(h)\}$. Then $\mathbb{Z}\vec{y} \subset H_x$ by Proposition 4, thus $p(x) = [H : H_x]$ is a finite number.
Furthermore, since $\vec{x}_1, \dots, \vec{x}_n$ generate H, we have for $x \notin \{x_1, \dots, x_n\}$ that $H_x = H$, which means $p(x) = 1$. $\square$

Proposition 6 (Serre Duality). *There is a dualizing element $\omega \in H$, that is, for each $\mathcal{F}$, $\mathcal{G} \in \operatorname{coh}^H(\mathbb{X})$ there is a natural isomorphism*

$$\mathrm{D}\operatorname{Ext}^1_{\mathbb{X}}(\mathcal{F}, \mathcal{G}) \simeq \operatorname{Hom}_{\mathbb{X}}(\mathcal{G}, \mathcal{F}(\omega)).$$

Here, D denotes the functor $\operatorname{Hom}_k(-, k)$.

Proof. The proofs of [4, III. 7] can be transferred to the graded situation, compare also [2]. $\square$

Remark. Let $x \in \mathbb{X}$ and $\mathcal{S}$ be a simple sheaf concentrated in x. By Serre duality it follows from Proposition 4 that $\mathcal{S}(\omega) \simeq \mathcal{S}(-\vec{x})$. Thus, $p(x) > 1$ is equivalent to $\operatorname{Ext}^1_{\mathbb{X}}(\mathcal{S}, \mathcal{S}) = 0$.

3. The Case of trivial Weights

Throughout this section we assume that R satisfies (F1)–(F5) and that $p(x_i) = 1$ for all $i = 1, \dots, n$. We then have $p(x) = 1$ for all $x \in \mathbb{X}$. We are going to show that R is isomorphic to the polynomial algebra $k[X, Y]$, graded by total degree.
Let K denote the function field of $\mathbb{X}$, which consists of all fractions $\frac{r}{s}$ where r, $s \in R$ are homogeneous, $s \neq 0$ and $\vec{r} = \vec{s}$. For each $\mathcal{F} \in \operatorname{coh}^H(\mathbb{X})$ define the rank $\operatorname{rk}(\mathcal{F}) = \dim_K(\mathcal{F}_\xi)_0$, where ξ denotes the generic point. Then $\operatorname{rk}(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ has finite length.
Let $\mathrm{K}_0(\mathbb{X})$ be the Grothendieck group of $\operatorname{coh}^H(\mathbb{X})$. Define linear forms χ, $\deg : \mathrm{K}_0(\mathbb{X}) \longrightarrow \mathbb{Z}$ by

$$\chi(\mathcal{F}) := \dim_k \operatorname{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{F}) - \dim_k \operatorname{Ext}^1_{\mathbb{X}}(\mathcal{O}, \mathcal{F})$$

(which is well-defined by Serre duality and since $\operatorname{Ext}^2_{\mathbb{X}}(-,-) = 0$) and

$$\deg(\mathcal{F}) := \chi(\mathcal{F}) - \operatorname{rk}(\mathcal{F}) \cdot \chi(\mathcal{O}_{\mathbb{X}})$$

for all $\mathcal{F} \in \operatorname{coh}^H(\mathbb{X})$.

Lemma 7. *Let $\mathcal{S}$ be a simple sheaf. Then* $\deg(\mathcal{S}) = 1$.

Proof. Let $\mathcal{S}$ be concentrated in $x = \mathfrak{p} \in \mathbb{X}$. Since $\deg(\mathcal{S}) = \chi(\mathcal{S})$ and $\operatorname{Ext}^1_{\mathbb{X}}(\mathcal{O}, \mathcal{S}) = 0$ by Serre duality it is sufficient to show that $\dim_k \operatorname{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{S}) = 1$. We have for open $V \subset \mathbb{X}$

$$\mathcal{S}(V) = \begin{cases} \mathcal{S}_x & x \in V, \\ 0 & x \notin V. \end{cases}$$

Let $f \in R$ homogeneous prime, which is not in $\mathfrak{p}$. Let $U := \mathbb{X}_f$. Then

$$\operatorname{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{S}) \simeq \operatorname{Hom}_{\mathcal{O}_{|U}}(\mathcal{O}_{|U}, \mathcal{S}_{|U}) \simeq \operatorname{Hom}_{R[f^{-1}]}(R[f^{-1}], S),$$

where S is a simple graded $R[f^{-1}]$-module. Since k is algebraically closed, the lemma follows by the following proposition. $\square$

Proposition 8 (Hilbert's Nullstellensatz). *Let $A = k[z_1, \dots, z_n]$ be a finitely generated k-algebra, which is graded by an abelian group H such that $k \subset A_0$ and $z_1, \dots, z_n$ are homogeneous. If A is a graded field, then $\dim_k A_0 < \infty$.*

Proof. The k-vector space A_0 is generated by the products $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ where $\sum_{i=1}^n \alpha_i \deg z_i = 0$. Since the kernel of the homomorphism $\mathbb{Z}^n \longrightarrow H$, $e_i \mapsto \deg z_i$ is finitely generated, so is the k-algebra A_0. Now the assertion follows by the ungraded Nullstellensatz. $\square$

By shifting the exact sequences $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(\vec{x}_i) \longrightarrow \mathcal{S}_i \longrightarrow 0$, where $\mathcal{S}_i$ is the simple sheaf concentrated in x_i, it follows inductively by the lemma that for all integers $\alpha_1, \dots, \alpha_n$ we have $\deg(\mathcal{O}(\sum_{i=1}^n \alpha_i \vec{x}_i)) = \sum_{i=1}^n \alpha_i$.
Hence there is a homomorphism $\delta : H \longrightarrow \mathbb{Z}$ with $\delta(h) = \deg(\mathcal{O}(h))$ for all $h \in H$. Now, if $\pi \in R_h \setminus \{0\}$, then π is prime if and only if $\delta(h) = 1$. Thus the set of homogeneous prime elements of R equals $\bigcup_{i=1}^n R_{\vec{x}_i} \setminus \{0\}$.

Proposition 9. *Assume $p(x_i) = 1$ for $i = 1, \dots, n$. Then R is isomorphic to $k[X, Y]$, graded by total degree.*

Proof. There are i, $j \in \{1, \dots, n\}$ with $i \neq j$ such that $\vec{x}_i = \vec{x}_j$. (Otherwise $\pi_1, \dots, \pi_n$ are up to association the only homogeneous prime elements; on the other hand, there is an $m > 0$ such that $m\vec{x}_1 = m\vec{x}_2$, thus we have infinitely many non-associated homogeneous elements $\pi_1^m - \lambda \pi_2^m$ ($\lambda \in k$) of the same degree, which gives a contradiction.) Without loss of generality let $\vec{x}_1 = \vec{x}_2$.
Then $S = k[\pi_1, \pi_2]$ is a graded Noether normalization of R, that is, S is graded regular and R a finitely generated graded S-module (since $R/(\pi_1, \pi_2)$ is graded artinian there is an $r > 0$ such that $(\pi_1, \dots, \pi_n)^r \subset (\pi_1, \pi_2)$). Therefore the induced morphism of schemes $f : \mathbb{X} \longrightarrow \mathbb{P}^1(k)$, $\mathfrak{p} \mapsto \mathfrak{p} \cap S$ is finite and hence closed.
In particular, for $i = 3, \dots, n$ we have $R\pi_i \cap S = S\pi_i'$, where $\pi_i' \in S$ is homogeneous prime, hence of the form $\alpha_i \pi_1 + \beta_i \pi_2$ (α_i, $\beta_i \in k$). Thus, we have for $i = 3, \dots, n$

$$\pi_i \sim \alpha_i \pi_1 + \beta_i \pi_2,$$

which proves the proposition. $\square$

4. Reduction of Weights

Assume that R satisfies conditions (F1)–(F5). Let $p_i = p(x_i) \geq 1$ be the number of simple sheaves which are concentrated in x_i ($i = 1, \ldots, n$). If $\mathcal{S}$ is a simple sheaf concentrated in x, let $H_x := \{h \in H \mid \mathcal{S} \simeq \mathcal{S}(h)\}$.

Lemma 10 (Reduction Lemma). *Assume that $p_1 > 1$. Let H' be the subgroup generated by $\vec{x}_2, \ldots, \vec{x}_n$, let $R' = k[\pi_1^{p_1}, \pi_2, \ldots, \pi_n]$. Then*
(1) $p_1 = \min\{l > 0 \mid l\vec{x}_1 \in H'\} = |H/H'|$. (2) $H' = H_{x_1}$. (3) $R' = R_{|H'}$.
(4) *The H'-graded algebra R' satisfies conditions (F1)–(F5).*
Let $\mathbb{X}' := \mathrm{Proj}^{H'}(R')$, $p' : \mathbb{X}' \longrightarrow \mathbb{N}$ be the weight function of $\mathbb{X}'$, let $x_1' \in \mathbb{X}'$ correspond to $\pi_1^{p_1}$ and let $x_i' \in \mathbb{X}'$ correspond to π_i ($i = 2, \ldots, n$). Then
(5) $p'(x_1') = 1$, $p'(x_i') = p_i$ *for* $i = 2, \ldots, n$.

Proof. (1), (2) Recall that $p_1 = [H : H_{x_1}]$. By Proposition 4 (4), $H' \subset H_{x_1}$. Therefore, $H_{x_1} \cap \mathbb{Z}\vec{x}_1 = \mathbb{Z}p_1\vec{x}_1$. Let $\sum_{i=1}^n \alpha_i \vec{x}_i \in H_{x_1}$. Then also $\alpha_1\vec{x}_1 \in H_{x_1}$, thus $\alpha_1 \in \mathbb{Z}p_1$. But $p_1\vec{x}_1 \in H'$ because there exists a homogeneous unit of degree $p_1\vec{x}_1$ in the localization R_{x_1}. Therefore, $H' = H_{x_1}$.
(3) By the same arguments.
(4) We have $\dim^{H'} R' = 2$, since $\dim^{H'} R' = \dim^H R'$ and R is finitely generated as H-graded R'-module.
(5) $p'(x_1') = 1$ is clear because of Proposition 4. Let $i \in \{2, \ldots, n\}$. It is enough to show that $H' \cap H_{x_i} = H'_{x_i'}$ because then $H'/H'_{x_i'} \simeq H' + H_{x_i}/H_{x_i} = H/H_{x_i}$. The inclusion $R'_{x_i'} \longrightarrow R_{x_i}$ shows $H' \cap H_{x_i} \supset H'_{x_i'}$. On the other hand we have $H' \cap H_{x_i} \subset H'_{x_i'}$ since obviously $R_{x_i|H'} = R'_{x_i'}$. □

Corollary 11. (1) $p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_n\vec{x}_n$. *Moreover, $H \simeq \mathbb{L}(\mathbf{p})$ as ordered groups, where $\mathbf{p} = (p_1, \ldots, p_n)$.*
(2) *There are pairwise different $\lambda_i \in k^*$ with $\pi_i^{p_i} \sim \pi_1^{p_1} - \lambda_i \pi_2^{p_2}$ ($i = 3, \ldots, n$).*

Proof. The subalgebra $k[\pi_1^{p_1}, \pi_2^{p_2}, \ldots, \pi_n^{p_n}]$ of R satisfies (F1)–(F5) and has only trivial weights. By the proof of Proposition 9 we have $p_1\vec{x}_1 = \cdots = p_n\vec{x}_n$. Because of Lemma 10 (1) these are the only relations. □

Proof of Theorem 1. Let us assume that $p_1 \geq p_2 \geq \cdots \geq p_s > 1$, $p_{s+1} = \cdots = p_n = 1$. Define $r = \max\{2, s\}$, $\mathbf{p} = (p_1, \ldots, p_r)$, $\underline{\lambda} = (\lambda_3, \ldots, \lambda_r)$. Then $H \simeq \mathbb{L}(\mathbf{p})$ is clear. Also $R = k[\pi_1, \ldots, \pi_r]$. Consider the epimorphism $\phi : k[X_1, \ldots, X_r] \longrightarrow R$, $X_i \mapsto \pi_i$. Because of Corollary 11 this induces an epimorphism $\overline{\phi} : \mathbb{S}(\mathbf{p}, \underline{\lambda}) \longrightarrow R$. Since R and $\mathbb{S}(\mathbf{p}, \underline{\lambda})$ have the same graded Krull dimension this is an isomorphism. □

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