Factorial Algebras, Quaternions and Preprojective Algebras

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1. Introduction

We establish a link between (graded) factoriality and the representation theory of finite dimensional algebras. For this link the preprojective algebra and a distinguished almost split sequence of coherent sheaves play a central role.

In more detail, let Λ be a tame hereditary k-algebra, and let L be an indecomposable preprojective Λ-module of defect -1. As shown in [**2**], functoriality of $\tau^- = \operatorname{Tr} \operatorname{D}$ induces on

$$\Pi(\Lambda, L) = \bigoplus_{n \geq 0} \operatorname{Hom}_\Lambda(L, \tau^{-n} L)$$

the structure of a $\mathbb{Z}$-graded k-algebra, which is finitely generated over k and has Krull dimension two. Moreover, $\Pi(\Lambda, L)$ depends only on the τ^--orbit of L and is said to be a *small preprojective algebra* of Λ.

If, in particular, Λ_F is the tame hereditary bimodule algebra $\begin{pmatrix} k & 0 \\ F & F \end{pmatrix}$ of a skew-field of quaternions F over k, there is exactly one Auslander-Reiten orbit of defect -1, hence a unique associated small preprojective algebra $\Pi(F)$, which is shown to be of the form $k[X, Y, Z]/Q_F$, where Q_F is an anisotropic ternary quadratic form over k (Theorem 2.1), and hence in particular $\Pi(F)$ is commutative and graded factorial.

Conversely, for an anisotropic ternary quadratic form Q over k we put $R_Q = k[X, Y, Z]/Q$ and denote by $\mathcal{O}$ the structure sheaf over the projective spectrum of R_Q. The almost split sequence $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}(1) \longrightarrow 0$ has an indecomposable middle term, called the *Auslander bundle*, whose endomorphism ring is a skew-field of quaternions F_Q over k (Theorem 2.1).

Our main result states that the correspondences $Q \mapsto F_Q$ and $F \mapsto Q_F$ are inverse to each other (Theorem 2.1).

2. Statement of Results

In this section we state the main results of the paper. The proofs, after some preparations in sections 3, 4 and 5, are given in the last section.

Throughout the article let k be a field of characteristic different from two. Let $Q = Q(X, Y, Z)$ be a ternary quadratic form over k which we assume to be non-degenerate. Recall that Q is called *anisotropic* if it has no non-trivial zero

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in k^3. Consider the $\mathbb{Z}$-graded k-algebra $R = k[X,Y,Z]/Q$, graded by total degree. The projective spectrum $\mathbb{X} = \operatorname{Proj}(R)$ of R is a non-singular one-dimensional scheme. Denote by $\operatorname{coh}(\mathbb{X})$ the category of coherent sheaves over $\mathbb{X}$, which we consider in the graded sense as in [**8**] and [**9**, §7]. Note, that the category of graded coherent sheaves is equivalent to the category of ungraded coherent sheaves since R is generated by elements of degree one. In $\operatorname{coh}(\mathbb{X})$ holds Serre duality $\operatorname{D}\operatorname{Ext}^1_{\mathbb{X}}(\mathcal{F},\mathcal{G}) \simeq \operatorname{Hom}_{\mathbb{X}}(\mathcal{G},\mathcal{F}(-1))$ functorially in $\mathcal{F}$, $\mathcal{G} \in \operatorname{coh}(\mathbb{X})$, where D is the duality $\operatorname{Hom}_k(-,k)$. It follows that $\operatorname{coh}(\mathbb{X})$ has almost split sequences and that the Auslander-Reiten translation is given by $\mathcal{F} \mapsto \mathcal{F}(-1)$. The middle term of the almost split sequence

$$\eta :\ 0 \longrightarrow \mathcal{O} \xrightarrow{\ u\ } \mathcal{A} \xrightarrow{\ v\ } \mathcal{O}(1) \longrightarrow 0 \tag{2.1}$$

starting in the structure sheaf $\mathcal{O}$ is called the *Auslander bundle*.

We recall that for $a,\, b \in k^* = k\setminus\{0\}$ the algebra of quaternions over k, denoted by $\left(\frac{a,b}{k}\right)$, is the k-algebra on two generators $\mathbf{i}$ and $\mathbf{j}$ with defining relations $\mathbf{i}^2 = a$, $\mathbf{j}^2 = b$, $\mathbf{ij} = -\mathbf{ji}$. It is well-known that $\left(\frac{a,b}{k}\right)$ is either a skew-field or isomorphic to the full matrix ring $\mathrm{M}_2(k)$ (see [**11**, Ch. 3,2.7]). Moreover, $\left(\frac{a,b}{k}\right)$ is a skew-field if and only if the associated norm form (of pure quaternions) $-aX^2 - bY^2 + abZ^2$ is anisotropic [**14**, Ch. 2, 11.10+11.14], and this provides a well-known bijection between the isoclasses of skew-fields of quaternions over k and the similarity classes of anisotropic ternary quadratic forms over k. Recall that two ternary quadratic forms Q_1 and Q_2 are called *similar* ($Q_1 \sim Q_2$), if there exists an invertible linear map $P : k^3 \longrightarrow k^3$ and an $\alpha \in k^*$, such that $Q_1(v) = \alpha Q_2(Pv)$ for all $v \in k^3$; it is easy to see that each non-degenerate ternary quadratic form over k is similar to a form $-aX^2 - bY^2 + abZ^2$, where $a,\, b \in k^*$. Theorem 2.1 will give a representation theoretical description of this bijection.

Let F be an algebra of quaternions over k. If F is a skew-field, let Λ_F be the tame hereditary k-algebra $\begin{pmatrix} k & 0 \\ F & F \end{pmatrix}$. If F is not a skew-field, that is $F \simeq \mathrm{M}_2(k)$, then let Λ_F be the Kronecker algebra $\begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}$ over k. In each of the two cases let P be the projective right Λ_F-module $(k\ 0)$ of defect -1 ([**5**]), and let

$$\Pi(F) = \Pi(\Lambda_F, P) = \bigoplus_{n\geq 0} \operatorname{Hom}_{\Lambda_F}(P, \tau^{-n}P)$$

be the small preprojective algebra attached to the Auslander-Reiten orbit of P (in the sense of [**2**]), where the multiplication on $\Pi(F)$ is defined by $fg = \tau^{-n}(f) \circ g$ (for all $f \in \operatorname{Hom}_{\Lambda_F}(P, \tau^{-m}P)$ and $g \in \operatorname{Hom}_{\Lambda_F}(P, \tau^{-n}P)$). For a different approach to preprojective algebras see [**6**].

THEOREM 2.1. (1) *If Q is an anisotropic ternary quadratic form over k and $\mathbb{X}$ the associated curve, then the endomorphism ring of the Auslander bundle $\mathcal{A}$ is a skew-field of quaternions over k.*

(2) *If $F \simeq \left(\frac{a,b}{k}\right)$ is a skew-field of quaternions over k, then the small preprojective algebra $\Pi(F)$ as a $\mathbb{Z}$-graded k-algebra is isomorphic to $k[X,Y,Z]/Q$, where $Q = -aX^2 - bY^2 + abZ^2$ is anisotropic over k.*

(3) *The assignments $Q \mapsto \operatorname{End}(\mathcal{A})$ and $F \mapsto \Pi(F)$ of (1) and (2) induce mutually inverse maps between the similarity classes of anisotropic ternary quadratic*

forms over k and the isomorphy classes of quaternion skew-fields over k, respectively.

The related correspondence between quaternion algebras and non-degenerate ternary quadratic forms is classical ([**14**, Ch. 2, 11.9], also [**15**], [**1**]), see also [**4**], [**3**] for a link to representation theory. Our result expresses this correspondence in terms of almost split sequences (Auslander bundle) and the small preprojective algebra.

A theorem of Samuel (see [**7**, 11.5]) states that R is factorial if and only if Q is anisotropic over k; moreover, in the isotropic case the divisor class group of R is of order two. Our next result, which is in part a representation theoretical variant of this theorem, provides a criterion for indecomposability of the Auslander bundle and establishes via tilting a link between ternary quadratic forms over k and the class of tame hereditary k-algebras defined above.

THEOREM 2.2. *Let Q be a non-degenerate ternary quadratic form over k. Then $\mathcal{O} \oplus \mathcal{A}$ is a tilting bundle in $\operatorname{coh}(\mathbb{X})$, where $\mathcal{A}$ is the Auslander bundle. Moreover*

(1) *$\mathcal{A}$ is indecomposable if and only if Q is anisotropic, and in this case the endomorphism ring of $\mathcal{O} \oplus \mathcal{A}$ is the tame hereditary algebra $\begin{pmatrix} k & 0 \\ F & F \end{pmatrix}$, where F is a skew-field of quaternions over k.*

(2) *If $\mathcal{A}$ is decomposable then there is a line bundle $\mathcal{L}$ such that $\mathcal{A} = \mathcal{L} \oplus \mathcal{L}$, and the endomorphism ring of the tilting bundle $\mathcal{O} \oplus \mathcal{L}$ is the Kronecker algebra over k.*

As application of Theorem 2.2 we get the following characterization of commutative small preprojective algebras. We recall that a tame bimodule ${}_GM_F$ is a bimodule such that F and G are finite dimensional skew-fields over k and k acts centrally on F, G and M such that $\dim {}_GM \cdot \dim M_F = 4$; the pair $(\dim {}_GM, \dim M_F)$ is called the *type* of M, compare [**5**]. For a $\mathbb{Z}$-graded algebra $R = \bigoplus_{n \in \mathbb{Z}} R_n$ we denote by $R^{(2)}$ the $\mathbb{Z}$-graded algebra $\bigoplus_{n \in \mathbb{Z}} R_{2n}$.

COROLLARY 2.3. *Let Λ be tame hereditary of the form $\begin{pmatrix} F & 0 \\ M & G \end{pmatrix}$, where $M = {}_GM_F$ is a tame bimodule, and let P be a projective Λ-module of defect -1. Then the small preprojective algebra $\Pi(P) = \Pi(\Lambda, P) = \bigoplus_{n \geq 0} \operatorname{Hom}_\Lambda(P, \tau^{-n}P)$ is commutative if and only if there is a field K such that as $\mathbb{Z}$-graded algebras either*

(1) *$\Pi(P) \simeq K[X,Y,Z]/Q$, where (a) $K \simeq F$, or (b) $K \simeq G$, and where Q is an anisotropic ternary quadratic form over K of the shape $-aX^2 - bY^2 + abZ^2$ (a, $b \in K^*$); moreover, in case (a) $G \simeq \left(\frac{a,b}{K}\right)$ and $M \simeq G$ is of type $(1,4)$, in case (b) $F \simeq \left(\frac{a,b}{K}\right)$ and $M \simeq F$ is of type $(4,1)$; or*

(2) *$\Pi(P) \simeq K[X,Y]^{(2)}$, where $K \simeq F \simeq G$, $M \simeq K \oplus K$ is of type $(2,2)$ and Λ is the Kronecker algebra over K.*

In our next result we characterize the $\mathbb{Z}$-graded algebras of the form $k[X,Y]$ and $k[X,Y,Z]/Q$, where Q is an anisotropic ternary quadratic form over k, in terms of group graded factorial algebras. Let H be a finitely generated abelian group of rank one equipped with a (partial) order, compatible with the group structure. Let R be an H-graded commutative k-algebra of graded Krull dimension two. Assume that R is affine over k (that is, a finitely generated k-algebra) and graded factorial, such that each homogeneous component R_h is of finite dimension over k, and $R_h \neq 0$ implies that h is positive in H; moreover, we assume that H is generated by those h

such that $R_h \neq 0$. We call such a k-algebra *H-graded factorial*. Let $\mathbb{X} = \mathrm{Proj}^H(R)$ be the graded projective spectrum of R (compare [**9**, §7]). Denote by $\mathrm{coh}^H(\mathbb{X})$ the category of graded coherent sheaves over $\mathbb{X}$ ([**9**, §7]). We call $\mathbb{X}$ (also R) *homogeneous*[1] if $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{S},\mathcal{S}) \neq 0$ for each simple sheaf $\mathcal{S} \in \mathrm{coh}^H(\mathbb{X})$. The number $g = \dim_k \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O},\mathcal{O})$ is the *genus* of $\mathbb{X}$ (or of R). Our following result extends [**10**, Prop. 9] (compare also [**13**]).

Theorem 2.4. *Let R be an H-graded factorial k-algebra of genus zero. If R is homogeneous, then $H \simeq \mathbb{Z}$, and as graded algebras either $R \simeq K[X,Y]$ or $R \simeq K[X,Y,Z]/Q$, graded by total degree, where K is a finite field extension of k and Q is an anisotropic quadratic form over K.*

3. The Auslander Bundle

Throughout this section let R be the k-algebra $k[X,Y,Z]/Q$, graded by total degree, where Q is a non-degenerate ternary quadratic form over k, and let $\mathbb{X}$ be its associated projective spectrum. Then R is a normal domain [**7**, 11.2]. Note that R is factorial (in the ungraded sense) if and only if it is graded factorial, since R is positively $\mathbb{Z}$-graded. Let $\mathbb{X}$ be the projective spectrum of R.

Define for $\mathcal{F}$, $\mathcal{G} \in \mathrm{coh}(\mathbb{X})$ the Euler form by

$$\langle \mathcal{F},\mathcal{G}\rangle = \dim_k \mathrm{Hom}(\mathcal{F},\mathcal{G}) - \dim_k \mathrm{Ext}^1(\mathcal{F},\mathcal{G}),$$

the rank by $\mathrm{rk}(\mathcal{F}) = \dim_K(\mathcal{F}_\xi)_0$ (where ξ denotes the generic point and K denotes the function field of $\mathbb{X}$, which consists of all fractions $\frac{r}{s}$ where r, $s \in R$ are homogeneous of the same degree, $s \neq 0$) and the degree by

$$\deg \mathcal{F} = \frac{1}{2}\cdot\langle \mathcal{O},\mathcal{F}\rangle - \frac{1}{2}\cdot \mathrm{rk}\,\mathcal{F};$$

we have $\deg \mathcal{F} \in \frac{1}{2}\mathbb{Z}$. We get the induced forms on the Grothendieck group $\mathrm{K}_0(\mathbb{X})$ of $\mathrm{coh}(\mathbb{X})$.

With regard to the slope $\mu = \frac{\deg}{\mathrm{rk}}$ we can define stability and semi-stability: Let $\mathcal{F} \in \mathrm{coh}(\mathbb{X})$ be a non-zero vector bundle (that is, a locally free sheaf). Then $\mathcal{F}$ is *stable* (*semi-stable*, resp.) if for each non-zero sub-bundle $\mathcal{G}$ with $\mathcal{G} \subsetneqq \mathcal{F}$ we have $\mu(\mathcal{G}) < \mu(\mathcal{F})$ ($\mu(\mathcal{G}) \leq \mu(\mathcal{F})$, resp.). Since the dualizing sheaf has degree -1, each indecomposable vector bundle $\mathcal{F}$ is stable (by the same arguments as in [**8**, Prop. 5.5]); in particular, $\mathcal{F}$ has a skew-field as endomorphism ring. It further follows that $\mathcal{F}$ has no self-extensions.

Let $\overline{k}$ be the algebraic closure of k. Let $\overline{R} = R \otimes_k \overline{k} = \overline{k}[X,Y,Z]/Q$ and $\overline{\mathbb{X}} = \mathrm{Proj}(\overline{R})$. Denote by $\overline{\cdot} : \mathrm{coh}(\mathbb{X}) \longrightarrow \mathrm{coh}(\overline{\mathbb{X}})$ extension of scalars given by $\mathcal{F} \mapsto \overline{\mathcal{F}} = \mathcal{F} \otimes_k \overline{k}$. We have a canonical isomorphism $\mathrm{Hom}_{\mathbb{X}}(\mathcal{F},\mathcal{G}) \otimes \overline{k} \longrightarrow \mathrm{Hom}_{\overline{\mathbb{X}}}(\overline{\mathcal{F}},\overline{\mathcal{G}})$ for all $\mathcal{F}$, $\mathcal{G} \in \mathrm{coh}(\mathbb{X})$.

We recall, that each four-dimensional central simple k-algebra is an algebra of quaternions over k, see [**14**, Ch. 8, 6.5].

Proposition 3.1. *Let $\mathcal{A}$ be the Auslander bundle on $\mathbb{X}$. Then the following holds:*

(1) *We have* $\mathrm{End}_{\mathbb{X}}(\mathcal{A}) \otimes_k \overline{k} \simeq \mathrm{End}_{\overline{\mathbb{X}}}(\overline{\mathcal{A}}) \simeq \mathrm{M}_2(\overline{k})$, *moreover* $\mathrm{End}_{\mathbb{X}}(\mathcal{A})$ *is an algebra of quaternions over* k.

[1] We use the representation-theoretical notion, compare [**5**].

(2) *If $\mathcal{A}$ decomposes then there is a line bundle $\mathcal{L} \in \mathrm{coh}(\mathbb{X})$ such that $\mathcal{A} = \mathcal{L} \oplus \mathcal{L}$ and $\mathrm{End}(\mathcal{L}) = k$. Moreover, each line bundle is isomorphic to $\mathcal{O}(n)$ or $\mathcal{L}(n)$ for some $n \in \mathbb{Z}$.*

(3) *The sheaf $\mathcal{A}$ is indecomposable if and only if R is graded factorial.*

PROOF. (1), (2) Scalar extension gives $\overline{R} \simeq \overline{k}[X,Y,Z]/(Z^2 - XY) \simeq \overline{k}[X,Y]^{(2)}$, and therefore $\mathrm{coh}(\overline{\mathbb{X}}) \simeq \mathrm{coh}(\mathbb{P}_1(\overline{k}))$. Moreover, scalar extension gives $\mathrm{End}_{\mathbb{X}}(\mathcal{A}) \otimes_k \overline{k} \simeq \mathrm{End}_{\overline{\mathbb{X}}}(\overline{\mathcal{A}})$ and applied to the sequence (2.1) yields an exact sequence

$$\overline{\eta}:\ 0 \longrightarrow \overline{\mathcal{O}} \longrightarrow \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{O}}(1) \longrightarrow 0$$

which does not split. Since $\dim_{\overline{k}} \mathrm{Ext}^1_{\overline{\mathbb{X}}}(\overline{\mathcal{O}}, \overline{\mathcal{O}}(1)) = 1$, we get that $\overline{\eta}$ is almost split. In $\mathrm{coh}(\mathbb{P}_1(\overline{k}))$ we have $\overline{\mathcal{A}} = \mathcal{N} \oplus \mathcal{N}$ for some line bundle $\mathcal{N} \in \mathrm{coh}(\mathbb{P}_1(\overline{k}))$. Therefore $\mathrm{End}(\overline{\mathcal{A}}) \simeq \mathrm{M}_2(\overline{k})$. The last assertion of (1) follows as in [**11**, Ch. 3, Prop. 1.1].

Assume that $\mathcal{A}$ decomposes. Then $\mathcal{A} \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles. Since $\overline{\mathcal{L}}_1 \simeq \overline{\mathcal{L}}_2$, we get $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \otimes \overline{k} \simeq \mathrm{Hom}(\overline{\mathcal{L}}_1, \overline{\mathcal{L}}_2) \neq 0$, and therefore $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \neq 0$. Scalar extension preserves rank and degree, in particular the slope μ. We get $\mu(\mathcal{L}_1) = \mu(\mathcal{L}_2)$, hence $\mathcal{L}_1 \simeq \mathcal{L}_2$ by stability.

If $\mathcal{M}$ is a line bundle in $\mathrm{coh}(\mathbb{X})$, then $\mu(\mathcal{M}) \in \frac{1}{2}\mathbb{Z}$, and $\overline{\mathcal{M}}$ is (over $\mathbb{P}_1(\overline{k})$) isomorphic to $\overline{\mathcal{O}}(n)$ or $\mathcal{N}(n)$ for some $n \in \mathbb{Z}$. The same argument as before proves $\mathcal{M} \simeq \mathcal{O}(n)$ or (if $\mathcal{A}$ decomposes) $\mathcal{M} \simeq \mathcal{L}(n)$ for some $n \in \mathbb{Z}$.

(3) Let $\mathcal{A}$ be indecomposable. By the arguments above each line bundle is isomorphic to $\mathcal{O}(n)$ for some $n \in \mathbb{Z}$, and hence R is graded factorial (compare [**13**, 1.10]). Conversely, if R is graded factorial and $\mathcal{A}$ decomposes, then $\mathcal{A} \simeq \mathcal{O}(n) \oplus \mathcal{O}(m)$, where $0 < n, m < 1$, contradiction. ☐

PROPOSITION 3.2. *The sheaf $\mathcal{T} = \mathcal{O} \oplus \mathcal{A}$ is a tilting bundle.*

PROOF. Since $\mu(\mathcal{O}) = 0$ and $\mu(\mathcal{A}) = \frac{1}{2}$ it follows that $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{T}, \mathcal{T}) = 0$ involving Serre duality and stability. Let $\mathcal{C}$ be a full subcategory of $\mathrm{coh}(\mathbb{X})$ which contains $\mathcal{O}$ and $\mathcal{A}$, and which is closed under the formation of direct summands, extensions, kernels of epimorphisms and cokernels of monomorphisms. It suffices to show that $\mathcal{C} = \mathrm{coh}(\mathbb{X})$. It is sufficient to prove that each line bundle and each simple sheaf lies in $\mathcal{C}$. Since $\mathcal{O}(1) \in \mathcal{C}$, we distinguish two cases:

(a) $\mathcal{A}$ is indecomposable. Then R is factorial, and each prime element π in R_1 induces a short exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\cdot\pi} \mathcal{O}(1) \longrightarrow \mathcal{S} \longrightarrow 0,$$

where $\mathcal{S}$ is a simple sheaf. Since $\mathcal{S}(n) \simeq \mathcal{S}$ (for each $n \in \mathbb{Z}$), each line bundle belongs to $\mathcal{C}$, therefore each simple sheaf, since those are cokernels of monomorphisms between line bundles.

(b) $\mathcal{A} = \mathcal{L} \oplus \mathcal{L}$. Then $\mathcal{L} \in \mathcal{C}$. There is a nonzero morphism $f \in \mathrm{Hom}(\mathcal{O}, \mathcal{L})$, and the cokernel $\mathcal{S}$ of f is simple (since f does not factor through a line bundle not isomorphic to $\mathcal{O}$ or $\mathcal{L}$), and $\mathcal{S} \in \mathcal{C}$. Since $\mathcal{S}(n) \simeq \mathcal{S}$ for all $n \in \mathbb{Z}$, it follows that $\mathcal{L}(1) \in \mathcal{C}$. Since also $\mathrm{Hom}(\mathcal{L}, \mathcal{O}(1)) \neq 0$, we get $\mathcal{O}(2) \in \mathcal{C}$, and inductively we see that each line bundle is contained in $\mathcal{C}$, and therefore also each simple sheaf. ☐

We note easy consequences of Proposition 3.2, where $\mathcal{E}$ is an indecomposable direct summand of $\mathcal{A}$ (hence $\mathcal{E} = \mathcal{A}$ or $\mathcal{E} = \mathcal{L}$).

COROLLARY 3.3. *The sheaf $\mathcal{O} \oplus \mathcal{E}$ is a tilting bundle.*

COROLLARY 3.4. *The Grothendieck group $\mathrm{K}_0(\mathbb{X})$ is equal to $\mathbb{Z}[\mathcal{O}] \oplus \mathbb{Z}[\mathcal{E}]$.*

COROLLARY 3.5. *Up to degree-shift each indecomposable vector bundle is isomorphic to $\mathcal{O}$ or $\mathcal{E}$.*

4. The Preprojective Algebra

PROPOSITION 4.1. *Let $R = k[X,Y,Z]/Q$, where Q is a non-degenerate ternary quadratic form over k, and let $F = F_Q$ be the endomorphism ring of the Auslander bundle $\mathcal{A}$. Then R and $\Pi(F)$ are isomorphic as $\mathbb{Z}$-graded algebras.*

PROOF. Let $\Lambda = \Lambda_F$. If $\mathcal{T} = \mathcal{O} \oplus \mathcal{E}$ is the tilting bundle from 3.3, then $\mathrm{End}(\mathcal{T}) = \Lambda$, $\mathrm{Hom}(\mathcal{T}, \mathcal{O}) = P$, and by tilting theory we get a full embedding

$$\mathrm{Hom}(\mathcal{T}, -) : \mathrm{add}\{\tau^{-n}\mathcal{T} \mid n \geq 0\} \longrightarrow \mathrm{mod}(\Lambda),$$

preserving Auslander-Reiten translations. Then

$$R \simeq \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(n)) \simeq \bigoplus_{n \geq 0} \mathrm{Hom}_{\Lambda}(P, \tau^{-n}P) = \Pi(F).$$

□

COROLLARY 4.2. *Let Q be a non-degenerate ternary quadratic form over k and $R = k[X,Y,Z]/Q$. Then R is graded factorial if and only if Q is anisotropic.*

PROOF. "$\Longrightarrow$" If Q is isotropic, then $R \simeq k[X,Y,Z]/(Z^2 - XY) \simeq k[X,Y]^{(2)}$, which is not graded factorial.

"$\Longleftarrow$" Assume that R is not graded factorial. If F is the endomorphism ring of the Auslander bundle, then Λ_F is the Kronecker algebra over k. By Proposition 4.1 we get $R \simeq k[X,Y]^{(2)} \simeq k[X,Y,Z]/(Z^2 - XY)$, thus $Q \sim Z^2 - XY$ is isotropic. □

Corollary 4.2 and Proposition 3.1 give a new proof of Samuel's theorem mentioned above. The next proposition is crucial for the proof of Theorem 2.1.

PROPOSITION 4.3. *Let $R = k[X,Y,Z]/Q$, where Q is a non-degenerate ternary quadratic form over k, and let $F = F_Q$ be the endomorphism ring of the Auslander bundle $\mathcal{A}$. If $Q \sim -aX^2 - bY^2 + abZ^2$, then $F \simeq \left(\frac{a,b}{k}\right)$.*

PROOF. Since the proof is trivial if Q is isotropic, we assume that Q is anisotropic. We know by Proposition 3.1 that F is an algebra of quaternions, let say $F \simeq \left(\frac{a',b'}{k}\right)$, $\mathbf{i}^2 = a'$, $\mathbf{j}^2 = b'$, $\mathbf{ij} = -\mathbf{ji}$. We consider $\mathbf{i}, \mathbf{j}$ also as endomorphisms of $\mathcal{A}$. Note that $\mathcal{A}$ is indecomposable. Recall that u and v are the morphisms in the sequence (2.1). Consider the almost split sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{f} \mathcal{O}(1)^{(4)} \xrightarrow{h} \mathcal{A}(1) \longrightarrow 0,$$

where f has components $f_1 = v$, $f_2 = v\mathbf{i}$, $f_3 = v\mathbf{j}$, $f_4 = v\mathbf{ij}$. Let $M = \mathrm{Hom}(\mathcal{A}, \mathcal{O}(1))$, $N = \mathrm{Hom}(\mathcal{O}(1), \mathcal{A}(1))$ and ${}^*M$ be the k-dual of M, and let ϕ_1, ϕ_2, ϕ_3, $\phi_4 \in {}^*M$ be the dual basis of f_1, f_2, f_3, $f_4 \in M$. Let $\sigma : {}^*M \longrightarrow N$ be the canonical embedding as in [**6**, Prop. 2.1]. By [**6**, Prop. 2.2] the components of h are $\sigma(\phi_1)$, $\sigma(\phi_2)$, $\sigma(\phi_3)$, $\sigma(\phi_4)$, and

$$\sum_{i=1}^{4} \sigma(\phi_i) f_i = 0.$$

Consider the basis $g_1 = u(1)$, $g_2 = \mathbf{i}(1)u(1)$, $g_3 = \mathbf{j}(1)u(1)$, $g_4 = \mathbf{ij}(1)u(1)$ (here, $f(1)$ denotes the morphism in $\mathrm{Hom}(\mathcal{F}(1), \mathcal{G}(1))$ associated to $f \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$; we sometimes write f instead of $f(1)$). Let $x = v\mathbf{i}u$, $y = v\mathbf{j}u$, $z = v\mathbf{ij}u \in$

$\mathrm{Hom}(\mathcal{O},\mathcal{O}(1)) = R_1$. We show $R = k[x,y,z] \simeq k[X,Y,Z]/(-a'X^2 - b'Y^2 + a'b'Z^2)$, from which $\left(\frac{a',b'}{k}\right) \simeq \left(\frac{a,b}{k}\right)$ and then the assertion will follow. It is sufficient to show that we have a relation (similar to) $-a'x^2 - b'y^2 + a'b'z^2 = 0$ (as morphisms we have $x^2 = x(1)x$ etc. by our convention). Let $\alpha_{ij} \in k$ such that $g_i = \sum_{j=1}^4 \alpha_{ij}\sigma(\phi_j)$. Then $\alpha_{ij} = \sigma^{-1}(g_i)(f_j)$. For all $f \in M$, $g \in N$, $h \in \mathrm{End}(\mathcal{A})$ we have the formula

$$\sigma^{-1}(h(1)g)(f) = \sigma^{-1}(g)(fh).$$

In the following we can assume that $\alpha_{11} = 1$. Therefore $A = (\alpha_{ij})$ and its inverse $A^{-1} = (\beta_{ij})$ are given by

$$A = \begin{pmatrix} 1 & \alpha & \beta & \gamma \\ \alpha & a' & -\gamma & -a'\beta \\ \beta & \gamma & b' & b'\alpha \\ \gamma & a'\beta & -b'\alpha & -a'b' \end{pmatrix}, \quad A^{-1} = \frac{1}{d}\begin{pmatrix} a'b' & -\alpha b' & -a'\beta & \gamma \\ -\alpha b' & b' & \gamma & -\beta \\ -a'\beta & -\gamma & a' & \alpha \\ \gamma & \beta & -\alpha & -1 \end{pmatrix}$$

with $d = a'b' - \alpha^2 b' - \beta^2 a' + \gamma^2 = \frac{1}{a'b'}|a'b'\mathbf{1} + \alpha b'\mathbf{i} + \beta a'\mathbf{j} + \gamma\mathbf{ij}| \neq 0$. We get

$$0 = d\cdot\sum_{i=1}^4 \sigma(\phi_i)f_i = v(1)\Bigg(\sum_{i,j=1}^4 d\beta_{ij}g_jf_i\Bigg)u \overset{(*)}{=} v(1)\Bigg(\sum_{i,j=2}^4 d\beta_{ij}g_jf_i\Bigg)u = b'x^2 + a'y^2 - z^2,$$

where equality $(*)$ follows from $vu = 0$. Now, multiplication with $a'b'$ proves our assertion. $\square$

5. Graded Factorial Algebras

In this section we prepare the proof of Theorem 2.4. Let R be an H-graded factorial k-algebra which is homogeneous. Let $\mathbb{X}$ be the H-graded projective spectrum of R and $g = \dim_k \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O},\mathcal{O})$ be the genus. Over an algebraically closed field each H-graded factorial algebra, which is homogeneous, is automatically of genus zero, as is shown in [**10**, Prop. 9]. It is open whether this also holds in this more general situation.

The assumptions on R imply that R_0 is a field, hence we can assume $R_0 = k$, and moreover, that $R = k[\pi_1,\dots,\pi_n]$, where the π_i are homogeneous prime. Note that each homogeneous unit in R has degree zero, since R is positively graded. Therefore we can define a further partial ordering $\leq$ on H by $h \leq h'$ if and only if $R_{h'-h} \neq 0$.

For $\mathcal{F} \in \mathrm{coh}^H(\mathbb{X})$ define $\deg' \mathcal{F} = \langle \mathcal{O},\mathcal{F}\rangle - (1-g)\cdot \mathrm{rk}\,\mathcal{F}$ and $\delta' : H \longrightarrow \mathbb{Z}$ by $\delta'(h) = \deg' \mathcal{O}(h)$ $(h \in H)$. It is easy to see that δ' is a homomorphism of groups. Let $\varepsilon = [\mathbb{Z} : \mathrm{Im}\,\delta']$. We have $\varepsilon = \min\{\langle \mathcal{O},\mathcal{S}\rangle \mid \mathcal{S} \text{ simple}\} = \min\{\dim_k \mathrm{End}\,\mathcal{S} \mid \mathcal{S} \text{ simple}\}$. Define $\deg = \deg'/\varepsilon$ and $\delta = \delta'/\varepsilon : H \longrightarrow \mathbb{Z}$. It is easy to see that $\mathrm{Ker}\,\delta$ is the torsion subgroup of H. Let $\mathcal{S}_0$ be simple such that $\deg \mathcal{S}_0 = 1$. Then $\mathrm{rk}\,\mathcal{F} = \frac{1}{\varepsilon}\langle \mathcal{F},\mathcal{S}_0\rangle$ for each $\mathcal{F} \in \mathrm{coh}^H(\mathbb{X})$.

We denote by $\omega \in H$ the dualizing element such that we have Serre duality $\mathrm{D}\,\mathrm{Ext}^1(\mathcal{F},\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{G},\mathcal{F}(\omega))$ functorially in $\mathcal{F}$, $\mathcal{G} \in \mathrm{coh}^H(\mathbb{X})$. Auslander-Reiten translation on $\mathrm{coh}^H(\mathbb{X})$ is given by $\mathcal{F} \mapsto \mathcal{F}(\omega)$. Denote by $\langle -,-\rangle$ the Euler form on the Grothendieck group $\mathrm{K}_0(\mathbb{X})$ of $\mathrm{coh}^H(\mathbb{X})$.

Proposition 5.1 (Riemann-Roch formula). *For all* $\mathbf{x}, \mathbf{y} \in \mathrm{K}_0(\mathbb{X})$ *we have*

$$\langle \mathbf{x},\mathbf{y}\rangle = (1-g)\cdot \mathrm{rk}\,\mathbf{x}\cdot \mathrm{rk}\,\mathbf{y} + \varepsilon\cdot \begin{vmatrix} \mathrm{rk}\,\mathbf{x} & \mathrm{rk}\,\mathbf{y} \\ \deg \mathbf{x} & \deg \mathbf{y} \end{vmatrix}.$$

PROOF. By graded factoriality each line bundle is of the form $\mathcal{O}(n)$ ($n \in \mathbb{Z}$). The formula is evidently valid if $\mathbf{x}$ is the class of a line bundle. Then the formula is also valid if $\mathbf{x}$ is the class of a simple sheaf, since each simple sheaf is cokernel of a monomorphism between line bundles. By non-singularity of $\mathbb{X}$ each vector bundle has a line bundle filtration, and so the formula holds in general. $\square$

We have two simple consequences of the Riemann-Roch formula.

COROLLARY 5.2. *If $g = 0$ then $\delta : H \longrightarrow \mathbb{Z}$ is an isomorphism.*

PROOF. Let h, $h' \in H$ such that $\delta(h) = \delta(h')$. Then by Riemann-Roch

$$\dim_k R_{h'-h} - \dim_k R_{\omega+h-h'} = \langle \mathcal{O}(h), \mathcal{O}(h') \rangle = 1 + \varepsilon(\delta(h') - \delta(h)) = 1.$$

$h \neq h'$ would imply $R_{h'-h} = 0$, hence $1 = -\dim_k R_{\omega+h-h'} \leq 0$, which is impossible. $\square$

COROLLARY 5.3. *If $g = 0$ then $\omega = -\frac{2}{\varepsilon}$; in particular $\varepsilon = 1$ or $\varepsilon = 2$.*

In the following let

$$\eta : 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}(-\omega) \longrightarrow 0$$

be almost split.

PROPOSITION 5.4. *Assume that the middle term $\mathcal{A}$ decomposes into two line bundles. Then $H \simeq \mathbb{Z}$, and $R \simeq k[X, Y]$, graded by total degree.*

PROOF. Because of factoriality we have $\mathcal{A} \simeq \mathcal{O}(h) \oplus \mathcal{O}(h')$ (h, $h' \in H$), and this implies $0 < h, h' < -\omega$. In particular $\omega < 0$, hence $g = 0$ and $H \simeq \mathbb{Z}$. Moreover, $\omega \leq -2$. Therefore, $\varepsilon = 1$ and $\omega = -2$. By Riemann-Roch we have $\dim_k R_n = n+1$ for $n \geq 0$. If π_1 and π_2 are linearly independent in R_1, then $R = k[\pi_1, \pi_2]$ easily follows by induction. $\square$

LEMMA 5.5. *If $g = 0$, then $R = k[R_1]$.*

PROOF. We can assume that $\mathcal{A}$ is indecomposable. Then $\varepsilon = 2$ ($\varepsilon = 1$ would imply $R \simeq k[X, Y]$ as above). Hence $\omega = -1$. It follows as in section 3 that $\mathcal{O} \oplus \mathcal{A}$ is a tilting bundle and that $\mathcal{O}$ and $\mathcal{A}$ are the only indecomposable vector bundles up to shift. Let $r \neq 0$ be a homogeneous element in R of degree n. This r corresponds to a morphism $f \in \operatorname{Hom}(\mathcal{O}, \mathcal{O}(n))$. Since η is almost split, and $\mathcal{O}(1)$ is the unique indecomposable summand of the middle term in the almost split sequence starting in $\mathcal{A}$, it follows that f is a sum of products of morphisms in $\operatorname{Hom}(\mathcal{O}(l), \mathcal{A}(l))$ and $\operatorname{Hom}(\mathcal{A}(l), \mathcal{O}(l+1))$ ($0 \leq l < n$). In particular f is a sum of products of morphisms in $\operatorname{Hom}(\mathcal{O}, \mathcal{O}(1))$, which proves the lemma. $\square$

6. Proofs of the Main Results

PROOF OF THEOREM 2.1. (1) This follows from Proposition 3.1 and Corollary 4.2.

(2) This follows from Proposition 4.1 and Proposition 4.3.

(3) Let Q_1 and Q_2 be ternary quadratic forms over k. The $\mathbb{Z}$-graded k-algebras $k[X, Y, Z]/Q_1$ and $k[X, Y, Z]/Q_2$ are isomorphic if and only if Q_1 and Q_2 are similar. Each (non-degenerate) ternary quadratic form is similar to one of the shape $-aX^2 - bY^2 + abZ^2$. Since two forms $-aX^2 - bY^2 + abZ^2$ and $-a'X^2 - b'Y^2 + a'b'Z^2$ are similar if and only if the algebras $\left(\frac{a,b}{k}\right)$ and $\left(\frac{a',b'}{k}\right)$ are isomorphic [**14**, Ch. 2, 11.9],

the assignments $Q \mapsto \mathrm{End}(\mathcal{A})$ and $F \mapsto \Pi(F)$ induce maps on the classes, and again by Proposition 4.1 and Proposition 4.3 these are inverse to each other. $\square$

Proof of Theorem 2.2. Follows from Proposition 3.1, Proposition 3.2 and Corollary 4.2. $\square$

Proof of Corollary 2.3. With the assumptions of Corollary 2.3, assume that $R := \Pi(P)$ is commutative. Let $K := \mathrm{End}(P) = R_0$. Then R is a commutative positively $\mathbb{Z}$-graded K-algebra. R is a finitely generated K-algebra: Since each non-zero morphism $f \in \mathrm{Hom}_\Lambda(P, \tau^{-n}P)$ $(n > 0)$ is a sum of compositions of irreducible morphisms between indecomposable right Λ-modules, which are necessarily preprojective, we get $R = K[R_1]$ by the special shape of the preprojective component of Λ, and R_1 is three-dimensional over K. Since moreover each non-zero morphism between preprojective Λ-modules of defect -1 is a monomorphism, R is graded integral. Let $\mathbb{X} = \mathrm{Proj}(R)$ be the projective spectrum of R. By [**12**, 4.10] and [**9**, 8.1] R is graded Cohen-Macaulay, and by [**9**, 7.3]

$$\mathrm{coh}(\mathbb{X}) \simeq \frac{\mathrm{mod}^{\mathbb{Z}}(R)}{\mathrm{mod}_0^{\mathbb{Z}}(R)} \simeq \frac{\mathrm{mod}^{\mathbb{Z}}(\Pi(\Lambda))}{\mathrm{mod}_0^{\mathbb{Z}}(\Pi(\Lambda))},$$

where $\Pi(\Lambda) = \bigoplus_{n \geq 0} \mathrm{Hom}_\Lambda(\Lambda, \tau^{-n}\Lambda)$ is the preprojective algebra attached to the Auslander-Reiten orbit of Λ ([**2**]). This also shows that

(1) if M is of type $(1,4)$ or $(4,1)$ then the line bundles in $\mathrm{coh}(\mathbb{X})$ are just the bundles of the form $\mathcal{O}(n)$ $(n \in \mathbb{Z})$;
(2) if M is of type $(2,2)$ then there are two degree-shift classes of line bundles in $\mathrm{coh}(\mathbb{X})$.

With [**2**] it follows that R is of graded Krull dimension two, and moreover, that $\mathrm{mod}^{\mathbb{Z}}(R_x)$ is a hereditary category, where R_x denotes the graded localization of R in a closed point $x \in \mathbb{X}$. Therefore each localization R_x is graded Dedekind, hence a graded normal domain. This implies that R itself is a graded normal domain (compare [**7**, 4.1]).

Since $\dim_K R_1 = 3$ and $\dim_K R_2 = 5$, we get $R \simeq K[X,Y,Z]/Q$, where Q is a quadratic form over K (see also the proof of Theorem 2.4 below). Since R is graded normal, we get that Q is non-degenerate. In case (1) the graded factoriality follows, that is, Q is anisotropic. In case (2) R is not graded factorial, hence Q is isotropic, thus $R \simeq K[X,Y,Z]/(Z^2 - XY) \simeq K[X,Y]^{(2)}$. Now the corollary follows from Theorem 2.1 and Theorem 2.2. $\square$

Proof of Theorem 2.4. Let R be an H-graded factorial algebra of genus zero, which is homogeneous. By Proposition 5.4 we can assume that $\mathcal{A}$ (as defined in section 5) is indecomposable. Then $\varepsilon = 2$, and by Riemann-Roch's formula 5.1 we have $\dim_k R_n = 2n + 1$ for $n \geq 0$. If π_1, π_2, $\pi_3 \in R_1$ are linearly independent then $R = k[\pi_1, \pi_2, \pi_3]$ by Lemma 5.5. Consider the canonical epimorphism $k[X,Y,Z] \longrightarrow R$. Comparing Krull dimensions we see that the kernel is generated by an irreducible homogeneous polynomial f. Calculating the dimensions in the short exact sequence

$$0 \longrightarrow N \longrightarrow R_1 \otimes_k R_1 \longrightarrow R_2 \longrightarrow 0$$

we see that f is of degree two, hence is a quadratic form. By Corollary 4.2 f is anisotropic over k. $\square$

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