INDECOMPOSABLE MODULES FOR DOMESTIC CANONICAL ALGEBRAS

DIRK KUSSIN AND HAGEN MELTZER

Dedicated to Professor Claus Michael Ringel on the occasion of his 60th birthday

ABSTRACT. We show that – up to precisely one – each exceptional module over a domestic canonical algebra of quiver type over a field k can be represented by matrices whose entries are just 0 and 1. In case $\operatorname{char} k \neq 2$ we calculate the matrices of these representations explicitly.

1. INTRODUCTION

This paper deals with the explicit determination of indecomposable modules over an important class of finite dimensional algebras. In representation theory of groups there is a long tradition in computing irreducible representations. In contrast to this, in representation theory of finite dimensional algebras there are only few cases where the problem of an explicit description of the indecomposable modules is solved. Gabriel described in [2] the indecomposable modules over the path algebra of a Dynkin quiver by explicit given matrices. Already in 1890 Kronecker [6] classified pairs of $n \times m$-matrices under the natural action of the group $\mathrm{GL}_n(k) \times \mathrm{GL}_m(k)$, in other words finite dimensional modules over the algebra which is called today the Kronecker algebra.

Nowadays it is modern to consider a module given by its dimension vector. In many cases one knows a lot about the general shape of the Auslander-Reiten quiver and has other important information, but it is difficult to describe the indecomposable modules *explicitly*, even in cases where one knows that there is, up to isomorphism, only one indecomposable module of a given dimension vector.

In particular, already for the class of extended Dynkin quivers the problem is difficult and unsolved in general. Nazarova [10] described the indecomposable modules for a special case, the 4-subspace problem, i.e. representations of the quiver $\widetilde{D}_4$ with subspace orientation. Surprisingly there are no explicit formulae for the matrices of the preprojective and preinjective representations for arbitrary extended Dynkin quivers.

On the other hand, Ringel has shown in [13] that for every finite dimensional path algebra A for *each* quiver (without relations) it is principally possible to describe the exceptional modules by matrices containing only 0 and 1 as entries. Recall that an A-module M is called *exceptional* if $\mathrm{End}(M)$ is a skew-field and $\mathrm{Ext}^i(M,M) = 0$ for all $i \geq 1$. A similar result for all indecomposable modules was proved by Dräxler in [1] for representation-finite algebras. But explicit lists of all exceptional modules in these cases are also not available.

Date: July 11, 2006.

2000 *Mathematics Subject Classification.* 16G20.

Key words and phrases. Canonical algebra, domestic type, exceptional module, preprojective module, zero-one matrices.

The second author was supported by the Polish Scientific Grant KBN 1 P03A 007 27.

In the present paper we study indecomposable modules for domestic canonical algebras of quiver type over a field k. This class of finite dimensional algebras was introduced by Ringel [11] and is related to the extended Dynkin quivers by tilting theory. Although the definition of canonical algebras is a little bit more complicated (they are path algebras of quivers *with relations*) the indecomposable modules are much easier to handle.

Let Λ be a domestic canonical algebra of quiver type. It follows already from [7, Cor. 5.3] that in case $\operatorname{char} k = 0$ each exceptional module can be exhibited using matrices with integer coefficients, or in case $\operatorname{char} k = p > 0$ with coefficients in $\mathbb{F}_p$. In this paper we show a much stronger result and give a general blueprint how to compute the matrices of the indecomposable modules, provided the characteristic of k is different from 2 and k is algebraically closed. The first assumption is needed only for the indecomposable modules of rank ± 6 for canonical algebras of type $(5, 3, 2)$ and the second assumption is used only for the regular modules.

Roughly speaking, these matrices for the preprojective and preinjective modules are obtained by "prolonging" the matrices of the corresponding Dynkin quiver (with subspace orientation) in a suitable way. Moreover, using a principle of "adding identities" we have to consider only very few cases. Altogether we obtain a nice principle for the computation of the matrices of all indecomposable modules in this situation. Concerning the coefficients of these matrices we prove in particular the following result.

Theorem 1. *Let k be a field and Λ a domestic canonical k-algebra of quiver type.*

(1) *Each preprojective and each preinjective indecomposable module can be exhibited by matrices involving as entries just* 0 *and* 1.

(2) *Each indecomposable module lying in an exceptional tube can be exhibited by matrices involving as entries just* 0, 1 *and* -1.

Using tilting theory and the explicit description of the indecomposable modules in the present paper it is possible to compute all indecomposable representations for the extended Dynkin quivers (under the assumption that the characteristic of k is different from 2 and k is algebraically closed). We will investigate this problem in a forthcoming paper.

A domestic canonical algebra of quiver type Λ is isomorphic to the path algebra of the quiver

(1.1)
$$\begin{tikzcd}
 & i_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{p-1}"] & i_{p-1} \arrow[dr, "\alpha_p"] & \\
0 \arrow[ur, "\alpha_1"] \arrow[r, "\beta_1"] \arrow[dr, "\gamma_1"'] & j_1 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{q-1}"] & j_{q-1} \arrow[r, "\beta_q"] & \infty \\
 & k_1 \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{s-1}"] & k_{s-1} \arrow[ur, "\gamma_s"] &
\end{tikzcd}$$

modulo the relation $\gamma_s \ldots \gamma_1 = \alpha_p \ldots \alpha_1 + \beta_q \ldots \beta_1$, where p, q, s is the length of the upper (middle, lower, respectively) arm, and where moreover the triples (p, q, s) are given by $(p, q, 1)$ (where $p,\ q \geq 1$), $(p, 2, 2)$ (where $p \geq 2$), $(3, 3, 2)$, $(4, 3, 2)$ and $(5, 3, 2)$ [11]. (Of course, for type $(p, q, 1)$ the third arm and the relation are redundant. Nevertheless, we will use this rather uniform description for all types.) Note, that over a non-algebraically closed field there is a more general definition of canonical algebras [12].

A finite-dimensional Λ-module M consists of finite-dimensional vector spaces $M(i)$ for each point i of the quiver, and a linear map $f(\alpha)$ for each arrow $\alpha = \alpha_i$, β_j and γ_k, satisfying the relation

$$f(\gamma_s) \circ \ldots \circ f(\gamma_1) = f(\alpha_p) \circ \ldots \circ f(\alpha_1) + f(\beta_q) \circ \ldots \circ f(\beta_1). \tag{1.2}$$

The number $\operatorname{rk}(M) = \dim M(\infty) - \dim M(0)$ is called the *rank* of M. Then an indecomposable module of positive rank (negative rank, rank zero, respectively) is preprojective (preinjective, regular, respectively). It is known that for each indecomposable module M over a domestic canonical algebra $|\operatorname{rk}(M)| \leq 6$. The global structure of the module category is also well-known [11]: There is precisely one preprojective component, precisely one preinjective component. All indecomposable preprojective and preinjective modules are exceptional. Note that by [7] the endomorphism ring of each exceptional module is k. The indecomposable regular modules form tubes, almost all of them homogeneous. If k is algebraically closed, then the family of tubes is parametrized by the projective line over k.

A typical example of an indecomposable preprojective module M_n with dimension vector

$$\begin{bmatrix} & n+1 & n+2 & n+3 & n+4 & \\ n & & n+1 & n+3 & & n+5 \\ & & n+2 & & & \end{bmatrix}$$

for Λ of type $(5,3,2)$ is given by

$$\begin{tikzcd}
& k^{n+1} \arrow[r, "X_{n+2,n+1}"] & k^{n+2} \arrow[r, "X_{n+3,n+2}"] & k^{n+3} \arrow[r, "X_{n+4,n+3}"] & k^{n+4} \arrow[dr, "X_{n+5,n+4}"] & \\
M_n : \; k^n \arrow[ur, "X_{n+1,n}"] \arrow[rr, "Y_{n+1,n}"] \arrow[drr, "Y_{n+2,n}"'] & & k^{n+1} \arrow[r, "Y_{n+3,n+1}"] & k^{n+3} \arrow[rr, "Y_{n+5,n+3}"] & & k^{n+5} \\
& & k^{n+2} \arrow[urrr, "Z_{n+5,n+2}"'] & & &
\end{tikzcd}$$

where the matrices $X_{n+i+1,n+i}$ ($Y_{n+i+1,n+i}$, $Y_{n+i+2,n+i}$, respectively) are obtained by adding to the $(n+i)\times(n+i)$-identity matrix as last row (as first row, as first and second row, respectively) $n+i$ zeros. The matrix $Z_{n+5,n+2}$ is obtained by enlarging the matrix

$$Z_{5,2} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

by two diagonals of length n consisting of the entry 1:

$$Z_{n+5,n+2} = \left[\begin{array}{c|ccc} & 1 & & \\ Z_{52} & & \ddots & \\ \hline & 1 & & 1 \\ 0 & & \ddots & \\ & & & 1 \end{array}\right].$$

Note that for $n = 0$ the module M_0 is an indecomposable representation of the Dynkin quiver $\mathbb{E}_8$ with subspace orientation. All the preprojective modules we construct are of the same type, we start with an indecomposable representation of a Dynkin quiver and "enlarge" it to an indecomposable representation of the corresponding domestic canonical algebra. The preinjective Λ-modules can be obtained by transposing the matrices of the preprojective modules with the dual dimension vectors. The indecomposable regular Λ-modules are described in chapter 4.

We will usually denote matrices by capital letters and their entries by the corresponding double-indexed small letter. For example $A = (a_{ij})$. We also write $a_{ij} = A[i, j]$.

For a quiver Q we denote by Q_0 its set of vertices and by Q_1 its set of arrows.

Acknowledgement: The authors would like to thank H. Lenzing for interesting discussions on the subject and an anonymous referee for useful comments leading to an improvement of the presentation of the results.

2. The preprojective modules

2.1. Let n and i be natural numbers. Let I_n be the $n \times n$-identity matrix. Define

$$X_{n+i,n} = \left[\begin{array}{ccc} & I_n & \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array}\right], \quad Y_{n+i,n} = \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ \hline & I_n & \end{array}\right] \in \mathrm{M}_{n+i,n}(k),$$

both having i zero rows of length n. We sometimes denote these matrices by I_*, X_{**} and Y_{**}, respectively, if their format is clear from the context. If Z is some matrix, then we call n-th *enlargement* of Z the matrix

$$\left[\begin{array}{c|ccc} & 1 & & \\ Z & & \ddots & \\ \hline & 1 & & 1 \\ 0 & & \ddots & \\ & & & 1 \end{array}\right]$$

with entries 1 on two diagonals each of length n.

2.2 (Adding identities)**.** Let M be a module over a canonical algebra $\Lambda = kQ/I$ of type (p, q, s) (with the notations above). We construct a module $\overline{M}$ over a canonical algebra $\overline{\Lambda}$ of type $(p+1, q, s)$ by one of the following operations:

(i) Replace in Q the arrow $0 \xrightarrow{\alpha_1} i_1$ by $0 \xrightarrow{\delta} i^* \xrightarrow{\alpha_1} i_1$.

(ii) Replace in Q the subpath $i_{j-1} \xrightarrow{\alpha_j} i_j \xrightarrow{\alpha_{j+1}} i_{j+1}$ by $i_{j-1} \xrightarrow{\alpha_j} i_j \xrightarrow{\delta} i^* \xrightarrow{\alpha_{j+1}} i_{j+1}$.

(iii) Replace in Q the arrow $i_{p-1} \xrightarrow{\alpha_p} \infty$ by $i_{p-1} \xrightarrow{\alpha_p} i^* \xrightarrow{\delta} \infty$.

In this way we get a new quiver $\overline{Q}$ and an ideal of relations $\overline{I}$ where $\alpha_p \dots \alpha_1$ is replaced by the composition of all arrows in the first arm in $\overline{Q}$. Obviously $\overline{\Lambda} = k\overline{Q}/\overline{I}$ is again a canonical algebra. Define a $\overline{\Lambda}$-module $\overline{M}$ by

$$\overline{M}(i^*) = \begin{cases} M(0) & \text{in case (i)}, \\ M(i_j) & \text{in case (ii)}, \\ M(\infty) & \text{in case (iii)}, \end{cases}$$

$\overline{M}(\delta)$ is the identity, and $\overline{M}$ coincides with M elsewhere. We say that $\overline{M}$ is obtained from M by *adding an identity*.

A similar construction can be done in the second and in the third arm, and also in the case the length of an arm is 1, where we have two possibilities. Moreover, this procedure can be iterated.

2.3. Let $\Lambda = kQ/I$ be a canonical algebra. In the following considerations we use the fact that for a preprojective Λ-module M and each arrow $\alpha \in Q_1$ the linear map $M(\alpha)$ is a monomorphism ([4, Lemma 4.2]).

We call an indecomposable preprojective Λ-module M *reduced* if there are no two different points i and j lying in the same arm with $\dim_k M(i) = \dim_k M(j)$.

Proposition 2.4. *Each indecomposable preprojective module over a domestic canonical algebra Λ which is not reduced can be obtained from a* reduced *indecomposable preprojective module over a domestic canonical algebra of smaller type by successively adding identities.*

Proof. Let M be a non-reduced indecomposable preprojective module of the domestic canonical algebra $\Lambda = kQ/I$. Then there is an arrow $i \xrightarrow{\delta} j$ such that $\dim_k M(i) = \dim_k M(j)$. By 2.3 we infer that $M(\delta)$ is an isomorphism. We construct from M a module M' over a domestic canonical algebra Λ' of smaller type as follows:

If δ is an arrow then there is another arrow which is a predecessor or successor in an arm of Q. By replacing a subpath of Q of the form $i' \xrightarrow{\varepsilon} i \xrightarrow{\delta} j$ (resp. $i \xrightarrow{\delta} j \xrightarrow{\varepsilon} j'$) by $i' \xrightarrow{\varphi} j$ (resp. $i \xrightarrow{\varphi} j'$) we get a new quiver Q'. Replacing $\delta\varepsilon$ (resp. $\varepsilon\delta$) by φ in the corresponding relation we get an ideal I' of relations in kQ'. This yields a canonical algebra $\Lambda' = kQ'/I'$ of smaller type which is consequently also domestic. Moreover, we define $M'(\varphi) = M(\delta) \circ M(\varepsilon)$ (resp. $M'(\varphi) = M(\varepsilon) \circ M(\delta)$), and leave M unchanged elsewhere.

By this modification we obtain a Λ'-module M'. Since $M(\delta)$ is an isomorphism it follows immediately from the definition of morphisms of representations that $\mathrm{End}_{\Lambda'}(M') \simeq \mathrm{End}_{\Lambda}(M) = k$. Consequently M' is indecomposable as Λ'-module. From $\mathrm{rk}(M') = \mathrm{rk}(M)$ we infer that M' is preprojective.

Repeating this procedure (if necessary) we obtained a reduced indecomposable preprojective module M'' over a domestic canonical algebra Λ''. Now, by adding an identity for each removed arrow we obtain an indecomposable preprojective Λ-module N. By construction we have $N \simeq M$. $\square$

2.5. Denote by $\langle -, - \rangle$ the Euler form on the Grothendieck group $\mathrm{K}_0(\Lambda)$ of Λ which is given by

$$\langle \underline{\dim}(X), \underline{\dim}(Y) \rangle = \dim_k \mathrm{Hom}_\Lambda(X, Y) - \dim_k \mathrm{Ext}^1_\Lambda(X, Y) + \dim_k \mathrm{Ext}^2_\Lambda(X, Y)$$

for all Λ-modules X and Y. (Note that Λ has global dimension at most 2.) Note that $\mathrm{K}_0(\Lambda) \simeq \mathbb{Z}^{|Q_0|}$, the simple modules correspond to the elements of the standard basis of $\mathbb{Z}^{|Q_0|}$.

Recall that an element $\mathbf{x} \in \mathrm{K}_0(\Lambda)$ is called a root if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$. A root $\mathbf{x} = (x_i)_{i \in Q_0}$ is called positive if all entries $x_i \geq 0$, that is, $\mathbf{x}$ is the dimension vector of a module. The rank of modules induces a linear form on $\mathrm{K}_0(\Lambda)$.

We denote by $\mathbf{1}$ the vector $\mathbf{x} = (x_i)$ with $x_i = 1$ for all $i \in Q_0$. This defines an element of the radical (compare [8]).

We remark that exceptional Λ-modules are up to isomorphism uniquely determined by their dimension vectors (which follows from [5, Lemma 4.2] or [9, Prop. 4.4.1]).

We now show the following.

Proposition 2.6. *Let Λ be a domestic canonical algebra. The function $\underline{\dim}$ induces a bijection between the isomorphism classes of indecomposable preprojective Λ-modules and positive roots of positive rank.*

The proof of Proposition 2.6 will follow from a result proved by T. Hübner in his diploma thesis, for which we now give a more conceptual argument. Denote by $\mathbb{X}$ the weighted projective line [4] associated with Λ. The main result of [4] is that the hereditary category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves over $\mathbb{X}$ contains a tilting bundle T such that $\mathrm{End}_{\mathbb{X}}(T) \simeq \Lambda$, and hence $\mathrm{coh}(\mathbb{X})$ and $\mathrm{mod}(\Lambda)$ are derived equivalent. We identify $\mathrm{K}_0(\mathbb{X}) = \mathrm{K}_0(\Lambda)$. For more details we refer to [4].

Proposition 2.7. *Let $\mathbb{X}$ be a weighted projective line of domestic type. Forming the class $E \mapsto [E]$ induces a bijection between the isomorphism classes of indecomposable vector bundles over $\mathbb{X}$ and the roots of positive rank in $\mathrm{K}_0(\mathbb{X})$.*

Proof. Since $\mathbb{X}$ is of domestic type there is a derived equivalence between $\mathrm{coh}(\mathbb{X})$ and $\mathrm{mod}(H)$, where $H = k\Delta$ is the path algebra of an extended Dynkin quiver Δ ([4, 5.4.1]). Any positive root in $\mathrm{K}_0(H)$ is a real root by [3, Thm. 6.5] and hence it follows from Kac's theorem (see [3, Thm. 7.4]) that each positive root in $\mathrm{K}_0(H)$ is the dimension vector of a unique exceptional H-module. Thus there is a bijection between isomorphism classes of exceptional H-modules and roots of $\mathrm{K}_0(H) = \mathrm{K}_0(\mathbb{X})$ modulo sign, and the result follows easily. $\square$

Proof of Proposition 2.6. By 2.5 it is sufficient to show that $\underline{\dim}$ induces a surjective map. Decompose $T = T_1 \oplus \ldots \oplus T_n$ into a direct sum of (pairwise non-isomorphic) indecomposables T_i. Let v be a positive root of positive rank. By Proposition 2.7 there is an indecomposable vector bundle E such that $v = [E]$. Positivity of v means $\langle T_i, v\rangle \geq 0$ and hence $\langle T_i, E\rangle \geq 0$ for $i = 1, \ldots, n$. Now it is not possible that *both* $\mathrm{Hom}_{\mathbb{X}}(T_i, E) \neq 0$ *and* $\mathrm{Ext}^1_{\mathbb{X}}(T_i, E) \neq 0$ hold, since (in the domestic case) by semistability [4] the first property would imply $\mu(T_i) \leq \mu(E)$, and the second property invoking Serre duality would give $\mu(E) \leq \mu(\tau T_i) < \mu(T_i)$, a contradiction. We conclude $\mathrm{Ext}^1_{\mathbb{X}}(T_i, E) = 0$ for $i = 1, \ldots, n$, and thus E is a Λ-module, necessarily indecomposable preprojective. $\square$

Corollary 2.8. *If M is an indecomposable preprojective Λ-module and $\mathbf{d} = \underline{\dim}\, M$ then for each $l = 1, \ldots, d_0$ there is an indecomposable preprojective Λ-module, unique up to isomorphism, with dimension vector $\mathbf{d}'$ such that $d'_i = d_i - l$ for each $i \in Q_0$, that is, $\mathbf{d}' = \mathbf{d} - l \cdot \mathbf{1}$.* $\square$

Proof. We apply Proposition 2.6. Since q is positive semidefinite, $\mathbf{d}$ is a root and $q(\mathbf{1}) = 0$ it follows easily that $\mathbf{d}' = \mathbf{d} - l \cdot \mathbf{1}$ is also a root. $\square$

2.9. It follows from the corollary and 2.3 that the indecomposable preprojective Λ-modules form series $(M_n)_{n\in\mathbb{N}}$ such that $M_0(0) = 0$ and $\underline{\dim}(M_n) = \underline{\dim}(M_0) + n\cdot\mathbf{1}$. Obviously, if M_0 is reduced then all M_n are also reduced. For each such series the module M_0 can be considered as indecomposable representation of the Dynkin quiver obtained from Q by deleting the vertex 0, and we can apply Gabriel's classification [2].

The dimension vectors $\mathbf{d}'$ of the reduced indecomposable preprojective modules M_0 which are of rank ≤ 5 are listed in Table 1. The matrix Z in the table is taken for defining $M_0(\gamma_2)$. The modules M_n arise from M_0 by enlarging the matrix Z as described in 2.1 and taking matrices X_{**} and Y_{**} of appropriate size for the other arrows.

In case of rank 6 the dimension vectors $\mathbf{d}'$ of the possible M_0 are the following

$$\begin{bmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{bmatrix}, \begin{bmatrix} & 1 & 2 & 3 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{bmatrix}, \begin{bmatrix} & 1 & 2 & 4 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{bmatrix},$$

$$\begin{bmatrix} & 1 & 3 & 4 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{bmatrix}, \begin{bmatrix} & 2 & 3 & 4 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{bmatrix}$$

Our main result is the following:

Theorem 2. *Let Λ be a domestic canonical algebra of quiver type. Let M be a reduced indecomposable preprojective Λ-module of rank r and with $\dim_k M(0) = n$.*

(1) *If $r = 1$, which only occurs for the Kronecker quiver, then M can be exhibited by the matrices $X_{n+1,n}$ and $Y_{n+1,n}$.*
(2) *If $2 \leq r \leq 5$ then M can be exhibited by matrices $M(\alpha_i) = X_{**}$ ($i = 1, \dots, p$), $M(\beta_j) = Y_{**}$ ($j = 1, \dots, q$), $M(\gamma_1) = Y_{**}$, and $M(\gamma_2)$ is the n-th enlargement of the matrix Z listed in Table 1 for $\mathbf{d}' = \mathbf{d} - n \cdot \mathbf{1}$.*
(3) *If $r = 6$ and $\operatorname{char} k \neq 2$ then M can be exhibited by matrices $M(\alpha_i) = X_{**}$ ($i = 1, \dots, p$), $M(\beta_j) = Y_{**}$ ($j = 1, \dots, q$), $M(\gamma_1) = Y_{**}$, and $M(\gamma_2)$ is the n-th enlargement of the matrix*

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\mathbf{d}'$	Z	$\mathbf{d}'$	Z
$0\ 1$	–	$\begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\begin{array}{cccc} & 1 & 2 & \\ 0 & 1 & 2 & 3 \\ & 1 & & \end{array}$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{array}{cccc} & 1 & 2 & \\ 0 & 1 & 2 & 3 \\ & 2 & & \end{array}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$
$\begin{array}{ccccc} & 1 & 2 & 3 & \\ 0 & 2 & & 3 & 4 \\ & & 2 & & \end{array}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{array}{ccccc} & 1 & 2 & 3 & \\ 0 & 1 & & 3 & 4 \\ & & 2 & & \end{array}$	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$
$\begin{array}{ccccc} & 1 & 2 & 3 & \\ 0 & 1 & & 2 & 4 \\ & & 2 & & \end{array}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$		
$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 0 & & 1 & 3 & & 5 \\ & & 2 & & & \end{array}$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 3 & & 5 \\ & & 2 & & & \end{array}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$
$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 0 & & 1 & 3 & & 5 \\ & & 3 & & & \end{array}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 4 & & 5 \\ & & 3 & & & \end{array}$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 3 & & 5 \\ & & 3 & & & \end{array}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 4 & & 5 \\ & & 2 & & & \end{array}$	$\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

TABLE 1. List of dimension vectors of the M_0 of rank ≤ 5

2.10. Together with Proposition 2.4 and Corollary 2.8 the preceding theorem gives a description of all indecomposable preprojective modules of rank ≤ 5 (and of all indecomposable preprojective modules under the assumption $\operatorname{char} k \neq 2$) for all domestic canonical algebras of quiver type.

If $\operatorname{char} k = 2$ then some of the modules of rank 6 constructed above with the matrix Z from Theorem 2 are decomposable (see Remark 3.3). However it is also true in characteristic 2 that each indecomposable preprojective module of rank 6 can be exhibited by 0-1-matrices, which is shown in [7, Corollary 5.3].

We stress the fact that all indecomposable preprojective modules of rank ≤ 5 over a domestic canonical algebra are constructed by enlarging Gabriel's representations of Dynkin quivers [2] and adding identities. However, in the case of rank 6 enlarging in the same way Gabriel's or Ringel's representations [13, Ch. 7] of the Dynkin quiver E_8 we get in general decomposable modules for the domestic canonical algebra. For example, the first enlargement of the indecomposable representation Q_1 from [2, Satz 4.4] is decomposable.

3. Proofs for the preprojective cases

In this section we give a proof of Theorem 2. It is sufficient to show that the representations defined by Theorem 2 have endomorphism ring k. The proof will be done by induction on $n = \dim M_0$. As illustration, for two of the rather difficult cases (one of rank 5 and one of rank 6) we give a detailed proof. The other cases are very similar or even easier (for smaller rank).

Assume first that $r \leq 5$. It is obvious that each module M with dimension vector $\mathbf{d}$ defined by the theorems gives rise to a module M' with dimension vector $\mathbf{d} - \mathbf{1}$ by dropping the last columns and rows in the matrices. Note that dropping the last column and row in matrices of the form X_{**} and Y_{**} gives again matrices of the form X_{**} and Y_{**}, respectively. The same is true for an enlargement of a matrix Z.

For $n = 0$, we deal with representations of the corresponding Dynkin quiver. It should be noted that the induction step is independent of the special form of the matrix Z.

3.1. For

$$M_n: \quad \begin{tikzcd} & k^{n+1} \arrow[r, "X_{n+2,n+1}"] & k^{n+2} \arrow[r, "X_{n+3,n+2}"] & k^{n+3} \arrow[r, "X_{n+4,n+3}"] & k^{n+4} \arrow[dr, "X_{n+6,n+4}"] & \\ k^n \arrow[ur, "X_{n+1,n}"] \arrow[rr, "Y_{n+1,n}"] \arrow[drr, "Y_{n+2,n}"'] & & k^{n+1} \arrow[r, "Y_{n+3,n+1}"] & k^{n+3} \arrow[rr, "Y_{n+5,n+3}"] & & k^{n+5} \\ & & k^{n+2} \arrow[urrr, "Z_{n+5,n+2}"'] & & & \end{tikzcd}$$

where $Z_{n+5,n+2}$ is the n-th enlargement of

$$Z_{52} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we have $\operatorname{End}(M_n) = k$.

Proof. Let

$$\begin{pmatrix} & B & C & D & E & \\ A & & F & G & & S \\ & & H & & & \end{pmatrix}$$

be an endomorphism of M_n, where $A \in \mathrm{M}_n(k), \dots, S \in \mathrm{M}_{n+5}(k)$. Dropping the indices we write $X = X_{**}$, $Y = Y_{**}$ and $Z = Z_{**}$. Thus we have the following commutativity relations:

$$BX = XA,\ CX = XB,\ DX = XC,\ EX = XD,\ SX = XE,$$
$$FY = YA,\ GY = YF,\ SY = YG,$$
$$HY = YA,\ SZ = ZH.$$

It follows, that A, B, C, D, E, F and G are nested submatrices of S. Furthermore, the shape of S is shown in Figure 1 where the gray shaded regions indicate zero lines. These follow by the relations for the first and the second arm. Note that for $n \geq 1$ the matrix A occurs twice, as left upper corner and as right lower corner. Therefore further entries are zero.

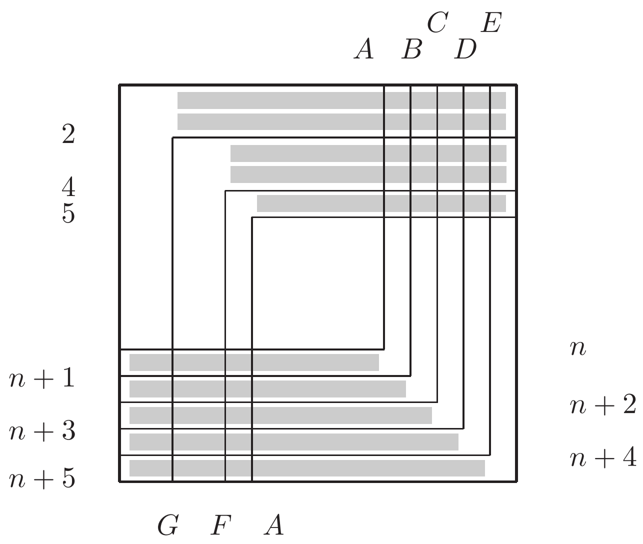

FIGURE 1. The shape of S

For $n = 0$, it follows that

$$S = \begin{bmatrix} s_{11} & s_{12} & 0 & 0 & 0 \\ 0 & s_{22} & 0 & 0 & 0 \\ 0 & 0 & s_{33} & s_{34} & 0 \\ 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & s_{55} \end{bmatrix}.$$

Exploiting the relation $SZ = ZH$, it follows immediately, that $s_{11} = s_{22} = s_{33} = s_{44} = s_{55}$ and $s_{12} = 0 = s_{34}$. Thus, $\mathrm{End}(M_0) = k$.

Now let $n \geq 1$ and assume that $\mathrm{End}(M_{n-1}) = k$. In the following, for a matrix U denote by U' the submatrix of U arising by deleting the last column and the last row from U. In order to apply the induction hypothesis we have to show the relations

$$B'X' = X'A',\ C'X' = X'B',\ D'X' = X'C',\ E'X' = X'D',\ S'X' = X'E', \tag{3.1}$$

$$F'Y' = Y'A',\ G'Y' = Y'F',\ S'Y' = Y'G', \tag{3.2}$$

$$H'Y' = Y'A',\ S'Z' = Z'H'. \tag{3.3}$$

This is true for the relations for the first arm, for example

$$SX = \boxed{\begin{array}{c|c} S' & \\ \hline & \end{array}} \cdot \boxed{\begin{array}{c|c} X' & \\ \hline 0 & \end{array}} = \boxed{\begin{array}{c|c} S'X' & \\ \hline & \end{array}}$$

By Figure 1 we have

$$e_{n+4,1} = \cdots = e_{n+4,n+3} = 0,$$

and because of $SX = XE$, this equals

$$XE = \boxed{\begin{array}{c|c} X' & \\ & 1 \\ \hline & 0 \end{array}} \cdot \boxed{\begin{array}{c|c} E' & \\ & \\ \hline 0 \ \cdots \ 0 & \end{array}} = \boxed{\begin{array}{c|c} X'E' & \\ \hline & \end{array}}$$

All other relations for the first arm follow by similar arguments. The relations for the second arm and $H'Y' = Y'A'$ follow easily, since Y is of the form

$$\boxed{\begin{array}{c|c} Y' & 0 \\ \hline 0 & 1 \end{array}}$$

We show $S'Z' = Z'H'$. We have

$$SZ = \boxed{\begin{array}{c|c} S' & \\ \hline & \end{array}} \cdot \boxed{\begin{array}{c|c} Z' & \\ \hline 0 & \end{array}} = \boxed{\begin{array}{c|c} S'Z' & \\ \hline & \end{array}}$$

and for

$$ZH = \boxed{\begin{array}{c|c} Z' & * \\ \hline 0 & 1 \end{array}} \cdot \boxed{\begin{array}{c|c} H' & \\ \hline * & \end{array}} = \boxed{\begin{array}{c|c} Z'H' & \\ \hline & \end{array}}$$

one has to show that

$$h_{n+2,1} = \cdots = h_{n+2,n+1} = 0.$$

In fact, for $j = 1, \dots, n+1$ we have

$$h_{n+2,j} = (Z \cdot H)[n+5, j] = (S \cdot Z)[n+5, j] = \sum_{k=1}^{n+5} s_{n+5,k} z_{kj} = 0,$$

since $s_{n+5,k} = 0$ for $k \leq n+4$ and $z_{n+5,j} = 0$. It follows, that all relations (3.1)-(3.3) are satisfied. By the induction hypothesis, all the matrices $A', \dots, S'$ are of the form αI_*, for the same $\alpha \in k$. Then the same follows for the matrices $A, \dots, S$. $\square$

Assume now that $r = 6$. By Remark 2.10 the usual induction step $n \to n+1$ does not work. Also, the steps $n \to n+l$ with $l \leq 5$ do not work. The reason is that in the first arm there is an arrow with dimension difference 2. Thus we use induction $n \to n+6$. This can be done for all types of dimensions vectors in the same manner. We illustrate this only in one case.

3.2. Assume that the characteristic of k is different from 2. For

$$\begin{tikzcd}
 & & k^{n+1} \arrow[r, "X_{n+2,n+1}"] & k^{n+2} \arrow[r, "X_{n+3,n+2}"] & k^{n+3} \arrow[r, "X_{n+4,n+3}"] & k^{n+4} \arrow[dr, "X_{n+6,n+4}"] & \\
M_n : & k^n \arrow[ur, "X_{n+1,n}"] \arrow[rr, "Y_{n+2,n}"] \arrow[drr, "Y_{n+3,n}"'] & & k^{n+2} \arrow[r, "Y_{n+4,n+2}"] & k^{n+4} \arrow[rr, "Y_{n+6,n+4}"] & & k^{n+6} \\
 & & & k^{n+3} \arrow[urrr, "Z_{n+6,n+3}"'] & & &
\end{tikzcd}$$

where $Z_{n+6,n+3}$ is the n-th enlargement of

$$Z_{63} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we have $\mathrm{End}(M_n) = k$.

Proof. Let

$$\begin{pmatrix} & B & C & D & E & \\ A & & F & G & & S \\ & & H & & & \end{pmatrix}$$

be an endomorphism of M_n, where $A \in \mathrm{M}_n(k), \ldots, S \in \mathrm{M}_{n+6}(k)$. As in the preceding proof, by the commutativity relations, A, B, C, D, E, F and G are nested submatrices of S. The shape of S is shown in Figure 2. As above, the gray shaded regions indicate zero lines. Here, the occurrence of $* = s_{n+6,n+5}$ makes the unusual induction step $n \to n+6$ necessary.

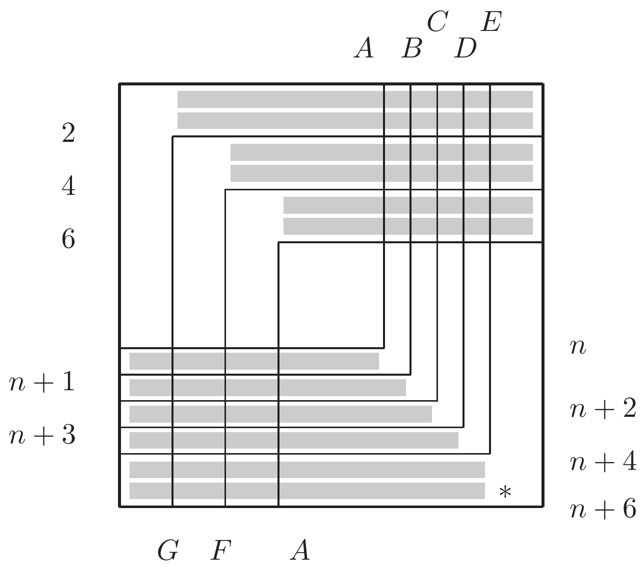

FIGURE 2. The shape of S

For $n = 0$ it follows that

$$S = \begin{bmatrix} s_{11} & s_{12} & 0 & 0 & 0 & 0 \\ 0 & s_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{33} & s_{34} & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{55} & s_{56} \\ 0 & 0 & 0 & 0 & s_{65} & s_{66} \end{bmatrix}.$$

The relation $SZ = ZH$ shows $s_{34} = 0 = s_{56}$, and moreover $s_{22} = s_{33} = s_{44} = s_{66}$, $s_{65} = s_{55} - s_{22} = -s_{12}$ and $s_{65} = s_{33} - s_{55}$. Then $2s_{65} = 0$. Since the characteristic is different from 2, we get the result. For the cases $n = 1, \ldots, 5$ the calculations are similar. We now make the induction step $n \to n + 6$.

Let $n \geq 6$ and assume that $\mathrm{End}(M_{n-6}) = k$. In the following, for a matrix U denote by U' the submatrix of U arising by deleting the last column and the last 6 columns and rows from U. Again we have to verify the relations (3.1)-(3.3).

For example, we have

$$SX = \left[\begin{array}{c|c} S' & \\ \hline & \end{array}\right] \cdot \left[\begin{array}{c|c} X' & \\ \hline 0 & \end{array}\right] = \left[\begin{array}{c|c} S' \cdot X' & \\ \hline & \end{array}\right]$$

and since by Figure 2 all

$$e_{n-1,1} = \cdots = e_{n-1,n-2} = 0 = e_{n,1} = \cdots = e_{n,n-2},$$

also

$$XE = \left[\begin{array}{c|cccc} X' & 1 & & & \\ & 0 & 1 & & \\ \hline & & & \ddots & \\ & & & & 1 \\ & & & & 0 \\ & & & & 0 \end{array}\right] \cdot \left[\begin{array}{ccc|c} & E' & & \\ \hline 0 & \cdots & 0 & \\ 0 & \cdots & 0 & \\ \hline & & & \end{array}\right] = \left[\begin{array}{c|c} X' \cdot E' & \\ \hline & \end{array}\right]$$

Then $SX = XE$ implies $S'X' = X'E'$.

All other relations for the first arm follow by similar arguments. The relations for the second arm and $H'Y' = Y'A'$ follow easily, since Y is of the form

$$\left[\begin{array}{c|c} Y' & 0 \\ \hline 0 & I_6 \end{array}\right]$$

In order to show $S'Z' = Z'H'$ we have to show

$$h_{ij} = 0 \text{ for } i = n-2, \ldots, n+3, \text{ and } j = 1, \ldots, n-3.$$

We have

$$h_{ij} = (Z \cdot H)[i+3, j] = (S \cdot Z)[i+3, j] = \sum_{k=1}^{n+6} s_{i+3,k} \cdot z_{kj}.$$

But $z_{kj} = 0$ for $k = n+1, \ldots, n+6$ and $s_{ik} = 0$ for $i = n+1, \ldots, n+6$ and $k = 1, \ldots, n$. Now the result follows as in the preceding proof. $\square$

Remark 3.3. We should note that 3.2 does not remain true in characteristic 2. In this case, even M_0 becomes decomposable.

4. The regular indecomposable modules

In order to complete our classification we describe in this section the regular indecomposable modules over domestic canonical algebras, provided k is algebraically closed. We study the more general case of a canonical algebra Λ of type (p,q,s) with arbitrary p,q,s. We assume here that $s>1$, for regular modules for path algebras of type $\tilde{A}_n$ we refer to [3, Chapter 11]. It is known from [13] that the Auslander-Reiten quiver of the regular Λ-modules consists of a family of orthogonal regular tubes with 3 exceptional tubes of rank p, q, s respectively while the other tubes are homogeneous.

Let

$$W_{n,n+1} = \left[\begin{array}{c|c} 0 & \\ \vdots & I_n \\ 0 & \end{array}\right] \in \mathrm{M}_{n,n+1}(k), \text{ and } J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & & \\ & & \ddots & \ddots & & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix} \in \mathrm{M}_n(k)$$

be the Jordan block matrix with eigenvalue λ. Recall also the definition of the matrix $X_{n+1,n}$ in 2.1 and of the quiver Q in (1.1).

4.1. We describe the indecomposable modules from the tube $\mathcal{T}$ of rank $p>1$ corresponding to the first arm of the canonical algebra $\Lambda = kQ/I$. For this we define for $1 \le a \le p$ and $\ell \in \mathbb{N}$ modules $S_a^{[\ell]}$. We write $\ell = np+t$ with $0 \le t < p$. There are the following cases.

(1) The case $1 \le a < p$.
   - (a) If $0 < t \le p-a$ the module $S_a^{[\ell]}$ is given by $S_a^{[\ell]}(x) = k^{n+1}$ for $x = i_a, i_{a+1}, \ldots, i_{a+t-1}$, and $S_a^{[\ell]}(x) = k^n$ for the remaining vertices, $S_a^{[\ell]}(\alpha_a) = X_{n+1,n}$, $S_a^{[\ell]}(\alpha_{a+t}) = W_{n,n+1}$, $S_a^{[\ell]}(\gamma_1) = J_n(1)$, all the linear maps for the remaining arrows are the identities;
   - (b) if $p-a<t<p$ then denote $t = p-a+s$. In this case the module $S_a^{[\ell]}$ is given by $S_a^{[\ell]}(x) = k^n$ for $x = i_s, i_{s+1}, \ldots, i_{a-1}$, and $S_a^{[\ell]}(x) = k^{n+1}$ for the remaining vertices, $S_a^{[\ell]}(\alpha_a) = X_{n+1,n}$, $S_a^{[\ell]}(\alpha_s) = W_{n,n+1}$, $S_a^{[\ell]}(\gamma_1) = J_{n+1}(1)$, all the linear maps for the remaining arrows are the identities;
   - (c) if $t=0$, that is $\ell = np$, we define $S_a^{[\ell]}$ by $S_a^{[\ell]}(x) = k^n$ for all $x \in Q_1$, $S_a^{[\ell]}(\alpha_a) = J_n(0)$, $S_a^{[\ell]}(\gamma_1) = J_n(1)$, all the linear maps for the remaining arrows are the identities.

(2) The case $a=p$.
   - (a) If $\ell = np+t$ with $0<t<p$ the module $S_p^{[\ell]}$ by $S_p^{[\ell]}(x) = k^n$ for $x = i_t, i_{t+1}, \ldots, i_{p-1}$, and $S_p^{[\ell]}(x) = k^{n+1}$ for the remaining vertices, $S_p^{[\ell]}(\alpha_p) = X_{n+1,n}$, $S_p^{[\ell]}(\alpha_t) = W_{n,n+1}$, $S_p^{[\ell]}(\gamma_1) = J_{n+1}(1)$, all the linear maps for the remaining arrows are the identities;
   - (b) if $\ell = np$ the modules $S_p^{[\ell]}$ are given by $S_p^{[\ell]}(x) = k^n$ for all $x \in Q_1$, $S_p^{[\ell]}(\alpha_p) = J_n(0)$, $S_p^{[\ell]}(\gamma_1) = J_n(1)$, all the linear maps for the remaining arrows are the identities.

Since $W_{n,n+1}X_{n+1,n} = J_n(0)$, $X_{n+1,n}W_{n,n+1} = J_{n+1}(0)$ and $J_n(0)+I_n = J_n(1)$, in all the cases $S_a^{(\ell)}$ defines a module, that is, the relation (1.2) is satisfied.

Proposition 4.2. *Let $\mathcal{T}$ be the tube of rank p corresponding to the first arm of Q. The $S_a^{[\ell]}$ (with $1 \le a \le p$ and $\ell \in \mathbb{N}$) are indecomposable, pairwise non-isomorphic modules in $\mathcal{T}$ of τ-order p. Moreover, $S_a^{[\ell]}$ is of length ℓ in $\mathcal{T}$ and has top $S_a^{[1]}$. In particular, the $S_a^{[1]}$ are the simple objects in $\mathcal{T}$.*

Proof. By similar methods as in section 3 one shows that the endomorphism ring of each $S_a^{[\ell]}$ is a local ring. We give the argument for the case (1)(a), the other cases are treated similarly. In this case an endomorphism of $S_a^{[\ell]}$ is given by two matrices $A \in \mathrm{M}_n(k)$ and $S \in \mathrm{M}_{n+1}(k)$ satisfying the following equations $X_{n+1,n}A = SX_{n+1,n}$, $W_{n,n+1}S = AW_{n,n+1}$ and $J_n(1)A = AJ_n(1)$. The first equation implies that A is a left upper corner of S, whereas the second equation yields that A is a right lower corner of S. In particular we infer that A is uniquely determined by S. Using $X_{n+1,n}W_{n,n+1} = J_{n+1}(0)$ we see that $SJ_{n+1}(0) = J_{n+1}(0)S$, and hence precisely the matrices S of the form $S = \sum_{i=0}^{n} \lambda_i J_{n+1}(0)^i$ (with $\lambda_0, \dots, \lambda_n \in k$) occur, which form the local endomorphism ring of the Jordan block matrix $J_{n+1}(0)$ considered as a representation of the one-loop quiver.

Comparing dimension vectors one sees that $S_a^{[\ell]}$ and $S_b^{[m]}$ for $(a, \ell) \ne (b, m)$ are non-isomorphic; for $\ell = m$ a multiple of p (but $a \ne b$) the non-isomorphy follows from the different positions of the non-isomorphism $J_n(0)$.

Obviously, for $1 \le a \le p-1$ the $S_a^{[1]}$ are the simple Λ-modules corresponding to the first arm, hence lying in $\mathcal{T}$. It follows that $S_p^{[1]}$ is also a simple object in $\mathcal{T}$, since the sum of the dimension vectors of all $S_a^{[1]}$ $(1 \le a \le p)$ is the radical vector $\mathbf{1}$.

For each $\ell \ge 1$ and each a with $1 \le a \le p$ there is an epimorphism $S_a^{[\ell+1]} \longrightarrow S_a^{[\ell]}$ (in particular, both indecomposables belong to the same tube). (This is easily verified by a case by case inspection involving the linear map $k^{n+1} \longrightarrow k^n$ defined by $W_{n,n+1}$.) Its kernel has the same dimension vector as a suitable simple object $S_b^{[1]}$, and hence is isomorphic to it. By induction, we assume that $S_a^{[\ell]}$ is of length ℓ with top $S_a^{[1]}$. We conclude that $S_a^{[\ell+1]}$ is an object of length $\ell + 1$ with the same top. This proves the proposition. □

We remark that an indecomposable module from a tube $\mathcal{T}$ of rank p is exceptional if and only if it is an object in $\mathcal{T}$ of length smaller then p. The modules of the tubes of rank q and s corresponding to the second and the third arm, respectively, of Q can be described in a similar way. Observe, however, that for modules of the rank s tube $\mathcal{T}'$ the coefficient -1 for the matrices is indispensable because the matrices have to satisfy the relations (1.2). In particular, let M be the simple object from $\mathcal{T}'$ with $M(0) = M(\infty) = k$ and $M(k_1) = \cdots = M(k_{s-1}) = 0$. Then one of the linear maps $M(\alpha_a)$ or $M(\beta_b)$ must be given by a negative number.

4.3. Finally we describe the regular Λ-modules belonging to all the homogeneous tubes, provided k is algebraically closed. For $\lambda \in k$, $\lambda \notin \{0, 1\}$ and $\ell \in \mathbb{N}$ we define a module $S_\lambda^{[\ell]}$ by $S_\lambda^{[\ell]}(x) = k^\ell$ for all vertices $x \in Q_0$, $S_\lambda^{[\ell]}(\alpha_1) = J_\ell(\lambda)$, $S_\lambda^{[\ell]}(\gamma_1) = J_\ell(\lambda + 1)$, and the identity maps for the remaining arrows.

Proposition 4.4. *Let $\lambda \in k$, $\lambda \ne 0, 1$. The modules $S_\lambda^{[\ell]}$ (where and $\ell \in \mathbb{N}$) are indecomposable objects of length ℓ in a homogeneous tube $\mathcal{T}_\lambda$.*

Proof. The statement follows straightforward by similar arguments as in 4.2. □

References

1. P. Dräxler, *Normal forms for representations of representation-finite algebras*, J. Symb. Comput. **32** (2001), no. 5, 491–497.
2. P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. **6** (1972), 71–103.
3. P. Gabriel and A. V. Roiter: *Representations of finite-dimensional algebras.* With a chapter by B. Keller. Encyclopaedia Math. Sci., 73, Algebra, VIII, 1–177, Springer, Berlin, 1992.
4. W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, Singularities, Representation of Algebras and Vector Bundles (Lambrecht 1985), Lecture Notes in Math., vol. 1273, Springer-Verlag, Berlin-Heidelberg-New York, 1987, pp. 265–297.
5. T. Hübner, *Exzeptionelle Vektorbündel und Reflektionen an Kippgarben über projektiven gewichteten Kurven*, Dissertation, Universität Paderborn, 1996.
6. L. Kronecker, *Algebraische Reduktion der Scharen bilinearer Formen*, Sitzungsber. Akad. Berlin (1890), 1225-1237.
7. D. Kussin and H. Meltzer, *The braid group action for exceptional curves*, Arch. Math. **79** (2002), 335–344.
8. H. Lenzing, *A K-theoretic study of canoncial algebras*, Representation Theory of Algebras (Cocoyoc, 1994), CMS Conf. Proc. vol. 18, Amer. Math. Soc., 1996, pp. 433–473.
9. H. Meltzer, *Exceptional vector bundles, tilting sheaves and tilting complexes for weighted projective lines*, Mem. Amer. Math. Soc. **171** (2004), no. 808, viii+139.
10. L. A. Nazarova, *Representations of a tetrad* (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 31 (1967), 1361–1378. English translation: Math. USSR, Izv. 1 (1967), 1305–1321.
11. C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Math., vol. 1099, Springer-Verlag, Berlin-Heidelberg-New York, 1984.
12. ______, *The canonical algebras*, Topics in Algebra, Banach Center Publ., no. 26, 1990, with an appendix by William Crawley-Boevey, pp. 407–432.
13. ______, *Exceptional modules are tree modules*, Linear Algebra Appl. **275-276** (1998), 471–493.

Institut für Mathematik, Universität Paderborn, 33095 Paderborn, Germany
E-mail address: dirk@math.upb.de

Instytut Matematyki, Uniwersytet Szczeciński, 70451 Szczecin, Poland
E-mail address: meltzer@wmf.univ.szczecin.pl