THE BRAID GROUP ACTION FOR EXCEPTIONAL CURVES

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Dedicated to Professor Idun Reiten on the occasion of her 60th birthday

ABSTRACT. We show that the operation of the braid group on the set of complete exceptional sequences in the category of coherent sheaves on an exceptional curve $\mathbb{X}$ over a field k is transitive. As a consequence the list of endomorphism skew-fields of the indecomposable direct summands of a tilting complex is a derived invariant. Furthermore, we apply the result in order to establish a bijection (which is compatible with the K-theory) between the sets of translation classes of exceptional objects in the derived categories of two derived-canonical algebras with the same Cartan matrix, but which are defined over possibly distinct fields.

1. INTRODUCTION

1.1. In the study of vector bundles on the complex projective plane $\mathbb{P}^2$ exceptional vector bundles play an important role. They were introduced by Drezet and Le Potier in connection with the investigation of stable bundles [4]. Rudakov showed in [23] that the braid group B_3 acts transitively on the set of triples of exceptional vector bundles on $\mathbb{P}^2$. Later this result was generalized, by Rudakov [24], to the product $\mathbb{P}^1 \times \mathbb{P}^1$ and, by Kuleshov and Orlov [10], to arbitrary del Pezzo surfaces. Moreover, in [1] Bondal developed the setting up of the braid group action on the set of exceptional sequences for an arbitrary triangulated k-category.

In the representation theory of algebras, it was shown by Crawley-Boevey [3] that the braid group B_n acts transitively on the set of complete exceptional sequences in the category of modules over a hereditary k-algebra, over an algebraically closed field k, with n simple modules. A simplification of the proof and a generalization to hereditary artin algebras was given by Ringel [22].

The braid group action on the set of complete exceptional sequences in a sheaf category $\mathrm{coh}(\mathbb{X})$ for a weighted projective line $\mathbb{X}$ in the sense of Geigle and Lenzing [5] was studied by the second author in [17]. It was shown that for a weighted projective line over an algebraically closed field this action is transitive, too. Applications to the study of tilting complexes were given in [18].

In this article we investigate the braid group action on the set of complete exceptional sequences for exceptional curves in the sense of Lenzing [15]. The main result is the following:

Theorem 1.1. *Let $\mathbb{X}$ be an exceptional curve over a field k and let n be the rank of the Grothendieck group of $\mathbb{X}$. Then the braid group B_n acts transitively on the set of complete exceptional sequences in the category of coherent sheaves* $\mathrm{coh}(\mathbb{X})$.

We note that the proof for weighted projective lines given in [17] cannot be extended to our general situation. The present proof basically exploits a rank formula due to Hübner.

From the fact that $\mathrm{coh}(\mathbb{X})$ admits a tilting sheaf whose endomorphism algebra is a canonical algebra in the sense of [21] we obtain straightforward

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Corollary 1.2. *let Λ be a canonical algebra over a field k with n simple modules. Then the semidirect product $\mathbb{Z}^n \ltimes B_n$ acts transitively on the set of complete exceptional sequences in the bounded derived category $\mathcal{D}^b(\Lambda)$ of* $\operatorname{mod}(\Lambda)$. □

Corollary 1.3. *For an exceptional curve $\mathbb{X}$ the list of endomorphism skew-fields of the indecomposable objects of a complete exceptional sequence in* $\operatorname{coh}(\mathbb{X})$ (*or of a tilting complex in the bounded derived category $\mathcal{D}^b(\mathbb{X})$ of* $\operatorname{coh}(\mathbb{X})$) *is a derived invariant.* □

We emphasize a very special case of this: Let Λ and Λ' be tubular canonical algebras which are derived-equivalent (see [11] for this phenomenon). Then $\Lambda/\operatorname{rad}\Lambda \cong \Lambda'/\operatorname{rad}\Lambda'$.

Corollary 1.4. *Let E be an exceptional sheaf on an exceptional curve. Then* $\operatorname{End}(E)$ *is isomorphic to the endomorphism ring of a simple module over the associated canonical algebra.* □

As further application we show that there is a bijection (which is compatible with the K-theory) between the sets of translation classes of exceptional objects in the derived categories of two derived-canonical algebras (possibly defined over distinct fields) with the same Cartan matrix.

2. Preliminaries

2.1. Throughout this paper k denotes a field. An object of a small abelian k-category $\mathcal{A}$ is called *exceptional* if the endomorphism ring of E is a skew-field and moreover $\operatorname{Ext}^i_{\mathcal{A}}(E,E) = 0$ for all $i > 0$. This notion extends in a natural way to the bounded derived category over $\mathcal{A}$, where the Ext-condition is replaced by the request $\operatorname{Hom}(E, E[j]) = 0$ for all $j \neq 0$. A sequence of exceptional objects $\epsilon = (E_1, \ldots, E_r)$ in $\mathcal{A}$ (respectively in $\mathcal{D}^b(\mathcal{A})$) is called an *exceptional sequence* if $\operatorname{Ext}^j(E_l, E_m) = 0$ (respectively $\operatorname{Hom}(E_l, E_m[j]) = 0$) for all $j \in \mathbb{Z}$ and all $l > m$. The exceptional sequence is called *complete* if the objects $E_1, \ldots, E_r$ generate the bounded derived category $\mathcal{D}^b(\mathcal{A})$ as triangulated category.

Following Lenzing [15, 2.5] an *exceptional curve* $\mathbb{X}$ is defined by the following requests on its associated category $\operatorname{coh}(\mathbb{X})$ of coherent sheaves:

1. $\operatorname{coh}(\mathbb{X})$ is a connected small abelian k-category with morphism spaces that are finite dimensional over k.
2. $\operatorname{coh}(\mathbb{X})$ is hereditary and noetherian and there exists an equivalence (called *Auslander-Reiten translation*) $\tau : \operatorname{coh}(\mathbb{X}) \to \operatorname{coh}(\mathbb{X})$ such that Serre duality
$$\mathrm{D}\operatorname{Ext}^1(X,Y) \cong \operatorname{Hom}(Y, \tau X)$$
holds.
3. $\operatorname{coh}(\mathbb{X})$ admits a complete exceptional sequence.

Note that condition 3. is equivalent to the existence of a tilting object in $\operatorname{coh}(\mathbb{X})$. Moreover, in case k is algebraically closed by [14] the exceptional curves are just the weighted projective lines. Also, in the algebraically closed case, Reiten and Van den Bergh [19] have recently classified the more general class of noetherian hereditary categories satisfying Serre duality, that is, categories satisfying conditions 1. and 2.

2.2. We recall some facts about exceptional curves, see [15, 16] for details. For an exceptional curve $\mathbb{X}$ we denote by $\operatorname{coh}_0(\mathbb{X})$ ($\operatorname{coh}_+(\mathbb{X})$, respectively) the subcategory of $\operatorname{coh}(\mathbb{X})$ of objects of finite length (of torsion-free objects, called (vector) *bundles*, respectively). Each indecomposable object lies either in $\operatorname{coh}_0(\mathbb{X})$ or in $\operatorname{coh}_+(\mathbb{X})$. The category $\operatorname{coh}_0(\mathbb{X})$ is a coproduct $\coprod_{x\in\mathbb{X}} \mathcal{U}_x$ of connected uniserial length categories.

For $x \in \mathbb{X}$ denote by $p(x)$ the number of (isomorphism classes of) simple objects in $\mathcal{U}_x$. Then $p(x) = 1$ for almost all x. Denote by x_i the points such that $p_i := p(x_i) > 1$ ($i = 1, \ldots, t$). We call $\mathbb{X}$ *homogeneous*, if $p(x) = 1$ for all $x \in \mathbb{X}$.

2.3. Let $\mathbb{X}$ be an exceptional curve and $\mathrm{K}_0(\mathbb{X})$ be the Grothendieck group of $\mathrm{coh}(\mathbb{X})$ which is equipped with the normalized Euler form $\langle -,- \rangle : \mathrm{K}_0(\mathbb{X}) \times \mathrm{K}_0(\mathbb{X}) \longrightarrow \mathbb{Z}$, which is defined on classes by

$$\langle [X],[Y] \rangle = \frac{1}{m}\big(\dim_k \mathrm{Hom}(X,Y) - \dim_k \mathrm{Ext}^1(X,Y)\big),$$

where m is some positive integer such that the resulting bilinear form is surjective. The Grothendieck group is a finitely generated free abelian group. The *Coxeter transformation* is an automorphism of $\mathrm{K}_0(\mathbb{X})$ which is induced by the Auslander-Reiten translation τ. Furthermore, the *radical* of $\mathrm{K}_0(\mathbb{X})$, that is the set of its fixed points, is an abelian group, which is finitely generated of rank one or two (see [13]).

There is a uniquely defined (normalized) *rank* function $\mathrm{rk} : \mathrm{K}_0(\mathbb{X}) \longrightarrow \mathbb{Z}$ (that is, a surjective linear map commuting with the Coxeter transformation) such that for each indecomposable F we have $\mathrm{rk}([F]) = 0$ if and only if $F \in \mathrm{coh}_0(\mathbb{X})$, and $\mathrm{rk}([F]) > 0$ for $F \in \mathrm{coh}_+(\mathbb{X})$. This determines a unique radical element $\mathbf{w} \in \mathrm{K}_0(\mathbb{X})$, which generates a rank one direct summand of the radical such that $\mathrm{rk} = \frac{1}{c}\langle -, \mathbf{w}\rangle$ for some positive integer c; actually, $c = [\mathbb{Z} : \langle \mathrm{K}_0(\mathbb{X}), \mathbf{w} \rangle]$.

There is a line bundle (that is, a bundle of rank one) C such that for each $i \in \{1,\dots,t\}$ there is exactly one simple sheaf $S_i \in \mathcal{U}_{x_i}$ with $\mathrm{Hom}(C,S_i) \neq 0$. Denote $\mathbf{a} := [C]$, $\mathbf{s_i} := [S_i]$ and define the numbers

$$\kappa := \langle \mathbf{a},\mathbf{a} \rangle,\ \varepsilon := \frac{\langle \mathbf{a},\mathbf{w} \rangle}{\langle \mathbf{a},\mathbf{a} \rangle},\ e_i := \frac{\langle \mathbf{a},\mathbf{s_i} \rangle}{\langle \mathbf{s_i},\mathbf{s_i} \rangle},\ f_i := \frac{1}{\varepsilon}\frac{\langle \mathbf{a},\mathbf{s_i} \rangle}{\langle \mathbf{a},\mathbf{a} \rangle}.$$

It follows that $c = \kappa\varepsilon$ and $\varepsilon \in \{1,2\}$. Let p be the least common multiple of the weights $p_1,\dots,p_t$. Then the *genus*, which is defined by

$$g(\mathbb{X}) := 1 + \frac{\varepsilon p}{2}\Big(\sum_{i=1}^{t} e_i f_i\Big(1 - \frac{1}{p_i}\Big) - \frac{2}{\varepsilon}\Big),$$

is an invariant of the Grothendieck group (compare [12]). The curve $\mathbb{X}$ is called *domestic* (*tubular*, *wild*, respectively) if $g(\mathbb{X}) < 1$ ($= 1$, > 1, respectively).

2.4. Let S be the direct sum of a τ-orbit of a simple sheaf. The middle term of the couniversal extension

$$0 \longrightarrow C^d \longrightarrow E \longrightarrow S \longrightarrow 0, \qquad d = [\mathrm{Ext}^1(S,C) : \mathrm{End}(C)]$$

is of the form $E = \overline{C}^r$, where $\overline{C}$ is an indecomposable bundle, independent of the choice of S; moreover, $\varepsilon = [\mathrm{Hom}(C,\overline{C}) : \mathrm{End}(C)]^{1/2}$ (see [15]). In [16, Prop. 5.4, Prop. 5.5]) it is shown, that there is a tilting bundle T in $\mathrm{coh}(\mathbb{X})$ of the following shape

(1)
$$\begin{tikzcd}
 & C_1(1) \arrow[r] & C_1(2) \arrow[r] & \cdots \arrow[r] & C_1(p_1-2) \arrow[r] & C_1(p_1-1) \arrow[dr] & \\
C \arrow[ur] \arrow[r] \arrow[ddr] & C_2(1) \arrow[r] & C_2(2) \arrow[r] & \cdots \arrow[r] & C_2(p_2-2) \arrow[r] & C_2(p_2-1) \arrow[r] & \overline{C} \\
 & \vdots & \vdots & & \vdots & \vdots & \\
 & C_t(1) \arrow[r] & C_t(2) \arrow[r] & \cdots \arrow[r] & C_t(p_t-2) \arrow[r] & C_t(p_t-1) \arrow[uur] &
\end{tikzcd}$$

such that the endomorphism algebra Λ of T is a canonical k-algebra in the sense of [21]. In particular, the bounded derived category $\mathcal{D}^b(\mathbb{X})$ of $\mathrm{coh}(\mathbb{X})$ is equivalent as triangulated category to the bounded derived category $\mathcal{D}^b(\Lambda)$ of $\mathrm{mod}(\Lambda)$. For $1 \leq i \leq t$ the $C_i(j)$ are given by the following couniversal exact sequences [16, Prop. 5.4, Prop. 5.5]

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^{\varepsilon f_i} & \longrightarrow & C_i(1) & \longrightarrow & \tau^{-1}S_i & \longrightarrow & 0 \\
0 & \longrightarrow & C_i(1) & \longrightarrow & C_i(2) & \longrightarrow & \tau^{-2}S_i & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & C_i(p_i-2) & \longrightarrow & C_i(p_i-1) & \longrightarrow & \tau^{-(p_i-1)}S_i & \longrightarrow & 0 \\
0 & \longrightarrow & C_i(p_i-1) & \longrightarrow & \overline{C}^{f_i} & \longrightarrow & S_i & \longrightarrow & 0.
\end{array} \tag{2}
$$

3. The braid group action on the set of exceptional sequences

3.1. We will apply the concept of mutations, developed in [7], [1] and [2] to the category $\mathrm{coh}(\mathbb{X})$ of coherent sheaves of an exceptional curve.

First, observe that as in the case of hereditary algebras or weighted projective lines in our situation the following lemma holds true (see [18]).

Lemma 3.1. *Let (E, F) be an exceptional pair in $\mathrm{coh}(\mathbb{X})$. Then we have $\mathrm{Hom}(E, F) = 0$ or $\mathrm{Ext}^1(E, F) = 0$.* □

3.2. For an exceptional pair (E, F) in $\mathcal{D}^b(\mathbb{X})$ we consider in the derived category of vector spaces the complex $\mathrm{Hom}^\bullet(E, F) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}(E, F[j])[-j]$ with trivial differential. Then the *left mutation* $\mathcal{L}_E F$ of F by E is defined by the distinguished triangle

$$(\Delta) \qquad \mathcal{L}_E F[-1] \longrightarrow \mathrm{Hom}^\bullet(E, F) \otimes_{\mathrm{End}(E)} E \xrightarrow{can} F \longrightarrow \mathcal{L}_E F$$

where *can* is the canonical morphism (compare [1]). Dually, the *right mutation* $\mathcal{R}_F E$ of E by F is defined by a triangle

$$\mathcal{R}_F E \longrightarrow E \xrightarrow{can} \mathrm{DHom}^\bullet(E, F) \otimes_{\mathrm{End}(F)} F \longrightarrow \mathcal{R}_F E[1].$$

Here $\mathrm{D} = \mathrm{Hom}_k(-, k)$ denotes the standard duality and duals of vector spaces have the reversed grading.

Lemma 3.2. *(i) $\mathrm{End}(\mathcal{L}_E F) \cong \mathrm{End}(F)$ and $(\mathcal{L}_E F, E)$ is an exceptional pair.*
(ii) $\mathrm{End}(\mathcal{R}_F E) \cong \mathrm{End}(E)$ and $(E, \mathcal{R}_E F)$ is an exceptional pair.

Proof. Applying the functor $\mathrm{Hom}(E, -)$ to the distinguished triangle (Δ) we obtain a long exact sequence
$\to \mathrm{Hom}(E, \mathcal{L}_E F[-1]) \to \mathrm{Hom}(\mathrm{Hom}^\bullet(E, F)\otimes_{\mathrm{End}(E)} E, E) \to \mathrm{Hom}(E, F) \to \mathrm{Hom}(E, \mathcal{L}_E F) \to$
which implies that $\mathrm{Hom}(E, \mathcal{L}_E F[i]) = 0$ for all $i \in \mathbb{Z}$. Further, the application of $\mathrm{Hom}(F, -)$ to (Δ) yields an isomorphism (of k-vector spaces) $\mathrm{Hom}(F, F) \xrightarrow{\cong} \mathrm{Hom}(F, \mathcal{L}_E F)$, and the application of $\mathrm{Hom}(-, \mathcal{L}_E F)$ to (Δ) yields an isomorphism $\mathrm{Hom}(\mathcal{L}_E F, \mathcal{L}_E F) \xrightarrow{\cong} \mathrm{Hom}(F, \mathcal{L}_E F)$. We define $\phi : \mathrm{Hom}(F, F) \longrightarrow \mathrm{Hom}(\mathcal{L}_E F, \mathcal{L}_E F)$, $\phi(f) = h$ where h satisfies $\gamma f = h\gamma$ with $\gamma : F \longrightarrow \mathcal{L}_E F$ the map in the triangle (Δ). It is easily checked that ϕ is an isomorphism of rings. This proves (i). The proof of (ii) is similar. □

The exceptional pairs $(\mathcal{L}_E F, E)$ and $(F, \mathcal{R}_F E)$ are called the *left* and *right mutation* of the pair (E, F), respectively.

3.3. Let B_r be the *braid group* on r strings, so with generators $\sigma_1, \dots, \sigma_{r-1}$ and with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $j \geq i+2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 1, \dots, r-2$. The group B_r acts on the set of exceptional sequences of length r in $\mathcal{D}^b(\mathbb{X})$ by

$$\sigma_i(E_1, \dots, E_r) = (E_1, \dots, E_{i-1}, E_{i+1}, \mathcal{R}_{E_{i+1}} E_i, E_{i+2}, \dots, E_r)$$
$$\sigma_i^{-1}(E_1, \dots, E_r) = (E_1, \dots, E_{i-1}, \mathcal{L}_{E_i} E_{i+1}, E_i, E_{i+2}, \dots, E_r).$$

Moreover, denote by G_r the semidirect product $\mathbb{Z}^r \ltimes B_r$ defined by the group homomorphism $B_r \to S_r \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^r)$ being the composition of the map, given by $\sigma_i \mapsto (i, i+1)$ (the transposition in the symmetric group S_r), with the natural action of S_r on $\mathbb{Z}^r$. Then also the group G_r acts on the set of exceptional sequences of length r in $\mathbb{X}$; for the elements e_i of the natural basis of $\mathbb{Z}^r$ we define

$$e_i(E_1, \ldots, E_r) = (E_1, \ldots, E_{i-1}, E_i[1], E_{i+1}, \ldots, E_r).$$

For any exceptional pair (E, F) in $\mathrm{coh}(\mathbb{X})$ there are uniquely determined exceptional sheaves $LF = L_E F$ and $RE = R_F E$ which, up to translation in the derived category, coincide with $\mathcal{L}_E F$ and $\mathcal{R}_F E$, respectively. Indeed, this is a consequence of the fact that $\mathcal{L}_E F$ and $\mathcal{R}_F E$ are exceptional in $\mathcal{D}^b(\mathbb{X})$. Furthermore, it follows from Lemma 3.1 that LF is defined by one of the three following exact sequences in $\mathrm{coh}(\mathbb{X})$

$$0 \longrightarrow LF \longrightarrow \mathrm{Hom}(E, F) \otimes_{\mathrm{End}(E)} E \xrightarrow{can} F \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(E, F) \otimes_{\mathrm{End}(E)} E \xrightarrow{can} F \longrightarrow LF \longrightarrow 0$$

$$0 \longrightarrow F \longrightarrow LF \longrightarrow \mathrm{Ext}^1(E, F) \otimes_{\mathrm{End}(E)} E \longrightarrow 0$$

and a similar fact is true for RE.

There is also an action of the braid group on the set of exceptional sequences of length r in the abelian category $\mathrm{coh}(\mathbb{X})$. In fact, it is easy to see that

$$\sigma_i(E_1, \ldots, E_r) = (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}} E_i, E_{i+2}, \ldots, E_r)$$

$$\sigma_i^{-1}(E_1, \ldots, E_r) = (E_1, \ldots, E_{i-1}, L_{E_i} E_{i+1}, E_i, E_{i+2}, \ldots, E_r)$$

define an action of B_r.

3.4. Similarly as in the case of hereditary algebras [3] and weighted projective lines [17], [18] one easily proves the following facts

Lemma 3.4. *Let $(E_1, \ldots, E_n)$ be a complete exceptional sequence in $\mathcal{D}^b(\mathbb{X})$. Then the smallest full triangulated subcategory $\langle E_1, \ldots, E_n \rangle$ containing all E_i coincides with $\mathcal{D}^b(\mathbb{X})$.* □

Lemma 3.5. *Any exceptional sequence $(E_1, \ldots, E_a, F_1, \ldots, F_c)$ in $\mathrm{coh}(\mathbb{X})$ can be enlarged to a complete exceptional sequence $(E_1, \ldots, E_a, H_1, \ldots, H_b, F_1, \ldots, F_c)$.* □

Lemma 3.6. *If $(E_1, \ldots, E_n)$ and $(F_1, \ldots, F_n)$ are complete exceptional sequences in $\mathrm{coh}(\mathbb{X})$ which differ in at most one place, say $E_j \cong F_j$ for $j \neq i$, then also $E_i \cong F_i$.* □

3.7. It is an interesting question whether, in addition to the braid group relations, there are other relations in the generators σ_i which are fulfilled in the action on the braid group operation. The following proposition shows that this is indeed the case in the tubular situation. A similar fact was proved for the braid group action on the set of complete exceptional sequences for hereditary algebras of finite representation type [25].

Proposition 3.7. *Let $\mathbb{X}$ be a tubular exceptional curve. Further, let n be the rank of the Grothendieck group of $\mathbb{X}$ and p the least common multiple of the weights $p_1, \ldots, p_t$. Then the element $(\sigma_{n-1} \cdot \ldots \cdot \sigma_2 \cdot \sigma_1)^{n \cdot p}$ acts trivially on the set of complete exceptional sequences in $\mathrm{coh}(\mathbb{X})$.*

Proof. Application of $(\sigma_{n-1} \cdot \ldots \cdot \sigma_2 \cdot \sigma_1)$ to an exceptional sequence $\epsilon = (E_1, \ldots, E_n)$ in $\mathrm{coh}(\mathbb{X})$ yields an exceptional sequence $(E_2, \ldots, E_n, F)$ for some $F \in \mathrm{coh}(\mathbb{X})$. Now, for each $i = 2, \ldots, n$, we have $\mathrm{Hom}(\tau^- E_1, E_i) \cong \mathrm{D}\mathrm{Ext}^1(E_i, E_1)$ and $\mathrm{Ext}^1(\tau^- E_1, E_i) \cong \mathrm{D}\mathrm{Hom}(E_i, E_1)$, by assumption, thus $(E_2, \ldots, E_n, \tau^- E_1)$ is an exceptional sequence. From Lemma 3.6 we conclude that $F \cong \tau^- E_1$. Therefore $(\sigma_{n-1} \cdot \ldots \cdot \sigma_2 \cdot \sigma_1)^n (E_1, \ldots, E_n) = (\tau^- E_1, \ldots, \tau^- E_n)$. Now, since $\mathbb{X}$ is tubular, the Coxeter transformation has finite order p. Because each exceptional sheaf is uniquely determined by its class in the Grothendieck group the assertion follows. □

4. The transitivity of the braid group action

4.1. In this section we will prove the main theorem. We emphasize that the proof given for weighted projective lines [17] cannot be adopted to the general situation. Indeed, that proof is based on [17, Lemma 3.1] which states that for an exceptional sequence $(E_1, E_2, \dots, E_n)$ in $\mathrm{coh}(\mathbb{X})$ such that $\dim \mathrm{Hom}(E_1, E_2) \geq 2$ and that $LE_2 = L_{E_1}E_2$ is defined by an exact sequence

$$0 \longrightarrow LE_2 \longrightarrow \mathrm{Hom}(E_1, E_2) \otimes E_1 \longrightarrow E_2 \longrightarrow 0$$

morphisms $0 \neq h \in \mathrm{Hom}(LE_2, E_1)$ and $0 \neq f \in \mathrm{Hom}(E_1, E_2)$ are either both monomorphisms or both epimorphisms. However in the general case, if for instance $(E_3, \dots, E_n)^{\perp}$ is equivalent to a tame bimodule algebra of type $(1, 4)$, then it occurs that f is a monomorphism and h is an epimorphism.

4.2. Our proof needs some preparation. Let $\mathbb{X}$ be an exceptional curve of arbitrary representation type and assume that the rank of the Grothendieck group $\mathrm{rk}(\mathrm{K}_0(\mathbb{X}))$ equals n.

Proposition 4.2. *Let E be an exceptional vector bundle in $\mathrm{coh}(\mathbb{X})$. Then the right perpendicular category $E^{\perp}$ is equivalent to a category of modules $\mathrm{mod}(H)$ of a hereditary (possibly non-connected) k-algebra H with $n-1$ simple modules. Moreover, denote by $P_1, \dots, P_{n-1}$ the indecomposable projectives in $E^{\perp}$. Then $\mathcal{P}_E := P_1 \oplus \cdots \oplus P_{n-1} \oplus E$ forms a tilting sheaf, and for each $1 \leq i \leq n-1$ we have $\mathrm{Hom}(P_i, E) \neq 0$.*

Proof. This was shown by Hübner [9, Theorem 5.4, Proposition 5.8] in the case that $\mathbb{X}$ is a weighted projective line over an algebraically closed field. It is easy to see that the proof carries over to our situation. □

Proposition 4.3. *Let E be an exceptional vector bundle in $\mathrm{coh}(\mathbb{X})$. Denote by $S_1, \dots, S_{n-1}$ a complete system of simple modules in $E^{\perp}$ and by $P_1, \dots, P_{n-1}$ their projective covers. Then*

$$\frac{\mathrm{rk}(E)^2}{\dim_k \mathrm{End}(E)} = \sum_{i=1}^{n-1} \frac{\mathrm{rk}(P_i) \cdot \mathrm{rk}(S_i)}{\dim_k \mathrm{End}(P_i)}.$$

Proof. We slightly generalize the original proof of Hübner [9, Proposition 5.9] in the algebraically closed case. By Proposition 4.2, $\mathcal{P}_E := P_1 \oplus \cdots \oplus P_{n-1} \oplus E$ defines a tilting sheaf with endomorphism ring $\Sigma := \mathrm{End}(\mathcal{P}_E)$. The simple Σ-module belonging to E is given by $\tau E[1]$. Denote $P_n := E$, $S_n := \tau E[1]$. For all $1 \leq i, j \leq n$ we have $\langle P_i, S_j \rangle = \delta_{ij} \cdot \dim_k \mathrm{End}(P_i)$. Since the classes of the P_i form a basis of $\mathrm{K}_0(\mathbb{X})$ we have

$$\mathrm{rk} = \langle -, \sum_{i=1}^{n} \frac{\mathrm{rk}(P_i) \cdot [S_i]}{\dim_k \mathrm{End}(P_i)} \rangle.$$

By non-degeneracy of $\langle -, - \rangle$ we conclude that $\mathbf{w} = \kappa\varepsilon \sum_{i=1}^{n} \frac{\mathrm{rk}(P_i) \cdot [S_i]}{\dim_k \mathrm{End}(P_i)}$. Because $\mathrm{rk}(\mathbf{w}) = 0$ we get the result. □

Lemma 4.4. *Let $\mathbb{X}$ be an exceptional curve, which is non-homogeneous. Then each complete exceptional sequence in $\mathrm{coh}(\mathbb{X})$ can be shifted by the braid group to an exceptional sequence which contains a simple sheaf.*

Proof. Since $\mathbb{X}$ is non-homogeneous we have $n > 2$. We first show that each orbit $\mathcal{O}$ under the braid group action admits an exceptional sequence which contains a finite length sheaf.

Assume contrary that there is an orbit $\mathcal{O}$ which does not have this property. From Lemma 3.2 we know that the sequence of endomorphism rings of an exceptional sequences including their multiplicities is, up to permutation, the same for each exceptional sequence of $\mathcal{O}$. Thus there is an exceptional vector bundle E appearing in an exceptional sequence ϵ in $\mathcal{O}$ such that $\frac{\mathrm{rk}(E)^2}{\dim_k \mathrm{End}(E)}$ is minimal. We can assume that ϵ is of the form $(E_1, \dots, E_{n-1}, E)$.

Now by Proposition 4.1, the right perpendicular category $E^{\perp}$ is equivalent to a module category $\mathrm{mod}(H)$ for a hereditary k-category H with $n-1$ simple modules. Denote by $(P_1,\dots,P_{n-1})$ a complete system of indecomposable projective modules and by $(S_{n-1},\dots,S_1)$ a complete system of simple modules in $E^{\perp}$, ordered in such a way that they form exceptional sequences. In particular one of the projective indecomposables is simple, say P_1 coincides with S_1. By [22], the braid group B_{n-1} acts transitively on the set of complete exceptional sequences in $E^{\perp}$. Thus there are braid group elements σ, $\sigma'\in B_{n-1}$ such that $\sigma(E_1,\dots,E_{n-1},E)=(P_1,\dots,P_{n-1},E)$ and $\sigma'(E_1,\dots,E_{n-1},E)=(S_{n-1},\dots,S_1,E)$.

Now, by Proposition 4.3

$$\frac{\mathrm{rk}(P_1)^2}{\dim_k \mathrm{End}(P_1)}+\sum_{i=2}^{n-1}\frac{\mathrm{rk}(P_i)\cdot\mathrm{rk}(S_i)}{\dim_k \mathrm{End}(P_i)}=\frac{\mathrm{rk}(E)^2}{\dim_k \mathrm{End}(E)}.$$

Since by assumption all $\mathrm{rk}(P_i)>0$ and all $\mathrm{rk}(S_i)>0$ we obtain

$$\frac{\mathrm{rk}(P_1)^2}{\dim_k \mathrm{End}(P_1)}<\frac{\mathrm{rk}(E)^2}{\dim_k \mathrm{End}(E)},$$

which gives a contradiction to the choice of E. Thus there is an exceptional sequence in $\mathcal{O}$ which contains a sheaf of finite length.

Then it follows as in [17, Lemma 3.3] that there is an exceptional sequence in $\mathcal{O}$ containing a simple sheaf. □

4.5. We now prove Theorem 1.1. If $n=2$, then $\mathbb{X}$ is homogeneous, hence each exceptional sequence is of the form $(\tau^i C,\tau^i\overline{C})$ or $(\tau^i\overline{C},\tau^{i-1}C)$ for $i\in\mathbb{Z}$, and the first pair is transformed to the second by the generator of $B_2\cong\mathbb{Z}$.

Now suppose that $n>2$. Let $\epsilon=(E_1,\dots,E_n)$ be an exceptional sequence in $\mathrm{coh}(\mathbb{X})$, and denote by $\mathbf{K}=(C,C_1(1),\dots,C_t(p_t-1),\overline{C})$ the exceptional sequence corresponding to the tilting bundle 2.4 (1). We have to show, that ϵ and $\mathbf{K}$ lie in the same orbit. By Lemma 4.4 we can assume without loss of generality, that $E_n=S$ is a simple sheaf, hence of the form $\tau^j S_i$ for some $i\in\{1,\dots,t\}$, $j\in\{0,\dots,p_i-1\}$.

From the exact sequences in 2.4 (2) we see that the left mutation of the pair $(C,C_i(1))$ equals $(\tau^{-1}S_i,C)$, the right mutation of $(C_i(p_i-1),\overline{C})$ equals $(\overline{C},S_i)$, and the right mutation of $(C_i(j),C_i(j+1))$ equals $(C_i(j+1),\tau^{-(j+1)}S_i)$, since $\varepsilon f_i=[\mathrm{Hom}(C,C_i(1)):\mathrm{End}(C)]$, $[\mathrm{Hom}(C_i(p_i-1),\overline{C}):\mathrm{End}(\overline{C})]=f_i$ and $1=[\mathrm{Hom}(C_i(j),C_i(j+1)):\mathrm{End}(C_i(j+1))]$ for $j=1,\dots,p_i-2$. Therefore, we see that $\mathbf{K}$ lies in the same orbit as some exceptional sequence $\epsilon'=(E'_1,\dots,E'_n)$ where $E'_n=S$. The right perpendicular category $S^{\perp}$ is of the form $\mathrm{coh}(\mathbb{X}')$ for some exceptional curve $\mathbb{X}'$ such that $\mathrm{K}_0(\mathbb{X}')$ is of rank $n-1$ (compare [6]). Hence we can apply the induction hypothesis which proves the theorem. □

5. Bijection between exceptional objects

5.1. For a derived-canonical algebra Λ denote by $\mathrm{Ex}(\Lambda)$ (respectively $\mathrm{Ex}^{\mathbb{Z}}(\mathcal{D}^b(\Lambda))$) the set of isomorphism classes (translation classes, respectively) of exceptional Λ-modules (objects in $\mathcal{D}^b(\Lambda)$, respectively). For $E\in\mathrm{Ex}^{\mathbb{Z}}(\mathcal{D}^b(\Lambda))$, the class in $\mathrm{K}_0(\Lambda)/\pm 1$ is also denoted by $[E]$. Let Λ and Λ' be two derived-canonical algebras (possibly defined over distinct base fields) with indecomposable projectives $P_1,\dots,P_n$ and $P'_1,\dots,P'_n$, respectively. Assume that (after permutation of the projectives) Λ and Λ' have the same Cartan matrices (associated to the classes of the projectives). Then there is an isomorphism $\sigma:\mathrm{K}_0(\Lambda)\longrightarrow\mathrm{K}_0(\Lambda')$ preserving the Euler form such that $\sigma[P_i]=[P'_i]$ for all i.

Theorem 5.1. *Let Λ and Λ' be derived-canonical algebras over the fields k and k', respectively, having the same Cartan matrix. Then there is a bijection $\phi:\mathrm{Ex}^{\mathbb{Z}}(\mathcal{D}^b(\Lambda))\longrightarrow\mathrm{Ex}^{\mathbb{Z}}(\mathcal{D}^b(\Lambda'))$ such that $[\phi(E)]=\sigma[E]$ for each E.*

Proof. Since exceptional objects are uniquely determined by their classes in the Grothendieck group there is an injective map $\iota : \mathrm{Ex}^{\mathbb{Z}}(\mathcal{D}^b(\Lambda)) \longrightarrow \mathrm{K}_0(\Lambda)/\pm 1$, and similarly an injective map ι' for Λ'. It is enough to show, that the isomorphism σ maps the image of ι into the image of ι'. Since mutations on the K-theoretical level are compatible with σ, this follows from the transitivity of the braid group action. □

5.2. We call a canonical algebra Λ *of quiver type*, if it is a canonical algebra in the classical sense [20], but over arbitrary base field.

Corollary 5.2. *Let Λ and Λ' be two canonical algebras of quiver type with same weight sequence. Then ϕ induces a bijection* $\mathrm{Ex}(\Lambda) \longrightarrow \mathrm{Ex}(\Lambda')$.

Proof. Let $\mathbb{X}$ and $\mathbb{X}'$ be associated exceptional curves to Λ and Λ', respectively, such that the projectives P_i and P_i' form tilting bundles as in 2.4 (1), respectively. Note that the P_i and P_i' are line bundles, since our algebras are of quiver type. Let $E \in \mathcal{D}^b(\Lambda)$ be exceptional. We show that E is *not* a translate of a Λ-module if and only if for its class $[E]$ in $\mathrm{K}_0(\Lambda)$ there are i, j such that $\langle [P_i], [E] \rangle > 0$ and $\langle [P_j], [E] \rangle < 0$. Assume without loss of generality, that E is a bundle in $\mathrm{coh}(\mathbb{X})$, and assume that E is not a translate of a Λ-module. Then there are i and j such that $\mathrm{Ext}^1(P_i, E) \neq 0$ and $\mathrm{Hom}(P_j, E) \neq 0$. Now, if $i = j$, then there is a non-zero map $\tau P_i \longrightarrow \tau E$, which is necessarily a monomorphism, and invoking Serre duality, there is a non-zero map $E \longrightarrow \tau P_i$. Composition of these maps gives $\mathrm{Ext}^1(E, E) \cong \mathrm{D}\mathrm{Hom}(E, \tau E) \neq 0$, which gives a contradiction since E is exceptional. Therefore, $i \neq j$, hence $\langle [P_i], [E] \rangle < 0$ and $\langle [P_j], [E] \rangle > 0$. By this characterization it follows, that ϕ maps translation classes of Λ-modules to those of Λ'-modules. □

Finally we formulate a special case:

Corollary 5.3. *Let Λ be a canonical k-algebra over an algebraically closed field k with parameters lying in the prime field Π (which always can be assumed for example, if Λ has at most three arms). Then each exceptional Λ-module is defined over Π.*

Proof. Let Λ' be the corresponding canonical Π-algebra of quiver type, and let E be an exceptional Λ'-module. Since the endomorphism ring of E is trivial, the same holds true for $\overline{E} = E \otimes_{\Pi} k$, hence $\overline{E}$ is exceptional. It follows that scalar extension induces a map between the isomorphism classes of exceptional modules over Λ' and Λ, respectively. By Corollary 5.2 this map is bijective. □

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