THE AUTOMORPHISM GROUPS OF DOMESTIC AND TUBULAR EXCEPTIONAL CURVES OVER THE REAL NUMBERS

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1. Introduction

This article is concerned with the geometry of the parametrizing sets $\mathbb{X}$ of separating tubular families of tame hereditary algebras and of canonical algebras of tubular type. Implicitly the geometrical structure of these parametrizing sets is completely known since they are just the exceptional curves introduced by Lenzing [13] (see also [14]). Such an exceptional curve $\mathbb{X}$ is defined via its associated category of coherent sheaves $\operatorname{coh}\mathbb{X}$. But an explicit description of the geometry over arbitrary base-fields is unknown. For algebraically closed base-fields an explicit description is possible and is given by the projective line with weighting (Geigle-Lenzing [5]). But for arbitrary base-fields this is difficult and unsolved.

The present article treats the case where the base-field is $k = \mathbb{R}$, the field of real numbers. The groundwork was laid by Dlab and Ringel who described the parametrizing sets explicitly as topological spaces [2, 16, 3, 4], the geometry was described partially also by Crawley-Boevey [1]. In [8, 9] a subclass of exceptional curves was described as projective spectrum of some explicit class of commutative graded factorial algebras. In spite of these works the problem was not completely solved. This will be done in the present paper. As we will see the description is essentially determined function-theoretically.

The geometry of $\mathbb{X}$ is basically described by its automorphism group, which is defined as the subgroup of the group of all isomorphism classes of auto-equivalences of the category $\operatorname{coh}\mathbb{X}$ which is generated by the auto-equivalences fixing the structure sheaf. It is surprising that these automorphisms also preserve metric structure of $\mathbb{X}$. Moreover, there is one case, where there exists a so-called ghost-automorphism of order two fixing all points of $\mathbb{X}$.

We describe an algebraic method to calculate the automorphism group of $\mathbb{X}$. The problem of the determination of the automorphism group is basically reduced to the homogeneous case. As we will see, in this case an exceptional curve is the Riemann sphere Σ, or some quotient $\Sigma/\mathbb{Z}_2$ of the Riemann sphere modulo an involution, and possibly equipped with additional structure (called colouring). We show that the automorphism group up to occurrence of ghost-automorphisms coincides with the group of all conformal maps on Σ, which are compatible with formation of the quotient and which preserve the additional structure.

Moreover, we determine the automorphism group of the derived category. We give lists of the automorphism groups in the domestic and tubular cases and discuss the occurrence of parameters in these cases.

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2. Calculation tools

Throughout this article let $k = \mathbb{R}$ be the field of real numbers. Let $\mathbb{X}$ be an exceptional curve, and denote by $\mathcal{H} = \operatorname{coh} \mathbb{X}$ the associated category of coherent sheaves. Let L be a line bundle in $\mathcal{H}$ (which is uniquely determined up to degree-shift, see below), which we call *structure sheaf*. Denote by $\operatorname{Aut} \mathcal{H} = \operatorname{Aut} \operatorname{coh}(\mathbb{X})$ the group of isomorphism classes of auto-equivalence of $\mathcal{H}$, called the automorphism group of $\mathcal{H}$ (more precisely: automorphism class group, but we use the shorter notion). The subgroup of $\operatorname{Aut} \mathcal{H}$ induced by those automorphisms fixing the structure sheaf L is denoted by $\operatorname{Aut} \mathbb{X}$ and called the automorphism group of $\mathbb{X}$.

Two exceptional curves $\mathbb{X}$ and $\mathbb{X}'$ are called *isomorphic* if there is an equivalence $\operatorname{coh} \mathbb{X} \longrightarrow \operatorname{coh} \mathbb{X}'$; we will see that in this case (over $k = \mathbb{R}$) there is even an equivalence sending the structure sheaf L of $\mathbb{X}$ to the structure sheaf L' of $\mathbb{X}'$.

Each exceptional curve arises by insertion of weights at finitely many points for some homogeneous exceptional curve [13]. The following proposition is not hard to prove (compare [15]).

Proposition 1. *Let $\overline{\mathbb{X}}$ be an exceptional curve with underlying homogeneous exceptional curve $\mathbb{X}$ such that $\overline{\mathbb{X}}$ arrises from $\mathbb{X}$ by insertion of weights $p_1, \ldots, p_t$ into the distinct points $x_1, \ldots, x_t$, respectively. Then $\operatorname{Aut} \overline{\mathbb{X}}$ can be identified with the subgroup of elements in $\operatorname{Aut} \mathbb{X}$ which preserve weights.*

Let $M = {}_F M_G$ be a bimodule over the skew-fields F and G, k acting centrally, with all data finite-dimensional over k. We always assume $M \neq 0$. Define the group $\operatorname{Aut} M = \operatorname{Aut}_k({}_F M_G)$ to be the set of all triples $(\varphi_F, \varphi_M, \varphi_G)$, where $\varphi_F \in \operatorname{Aut}_k(F)$, $\varphi_G \in \operatorname{Aut}_k(G)$, $\varphi_M : M \longrightarrow M$ is k-linear and bijective, and for all $f \in F$, $g \in G$ and $m \in M$ we have

$$\varphi_M(fmg) = \varphi_F(f)\varphi_M(m)\varphi_G(g).$$

Composition and inverse are built componentwise, the neutral element is given by $(1_F, 1_M, 1_G)$. Note, that projection onto the middle component, $(\varphi_F, \varphi_M, \varphi_G) \mapsto \varphi_M$ is injective. There is an alternative description: Consider the k-category consisting of two objects with endomorphism ring F and G, respectively, and with non-zero Hom-space only in one direction, which is given by M. Then an automorphism of the bimodule M is just a k-self-equivalence of this category.

An element $(\varphi_F, \varphi_M, \varphi_G) \in \operatorname{Aut} M$ is called *inner*, if there are $f \in F^*$, $g \in G^*$ such that for all $x \in F$, $y \in G$, $m \in M$ we have $\varphi_F(x) = f^{-1}xf$,

$\varphi_G(y) = g^{-1}yg$ and $\varphi_M(m) = f^{-1}mg$. The subgroup of all inner automorphisms is denoted by $\operatorname{Inn} M = \operatorname{Inn}_k({}_F M_G)$, the factor group by $\operatorname{Out} M = \operatorname{Out}_k({}_F M_G) = \operatorname{Aut} M / \operatorname{Inn} M$.

Each element $(\varphi_F, \varphi_M, \varphi_G) \in \operatorname{Aut} M$ defines a k-algebra automorphism on the hereditary algebra $\Lambda := \begin{pmatrix} G & 0 \\ M & F \end{pmatrix}$ in the obvious way; then, the triple is inner if and only if the induced k-algebra automorphism is inner.

Proposition 2. *Let $\mathbb{X}$ be a homogeneous exceptional curve with underlying tame bimodule $M = {}_F M_G$. Then*

$$\operatorname{Aut} \mathbb{X} \simeq \operatorname{Out} M.$$

Proof. Denote by $\overline{L}$ the indecomposable bundle such that there is an irreducible map from L to $\overline{L}$. Then $M = \operatorname{Hom}(L, \overline{L})$. Let φ be an auto-equivalence of $\mathcal{H} = \operatorname{coh} \mathbb{X}$ fixing the structure sheaf L. Then φ also fixes $\overline{L}$. Therefore, by restriction φ induces an auto-equivalence of the full subcategory $\{L, \overline{L}\}$, hence an element of $\operatorname{Aut} M$. Moreover, the functor φ is isomorphic to the identity if and only if the induced automorphism on the bimodule is inner.

Conversely, any element φ in $\operatorname{Aut} M$ induces an automorphism of the bimodule algebra Λ, hence gives an auto-equivalence of $\operatorname{mod}(\Lambda)$, hence also of $\mathrm{D}^b(\Lambda)$, and since $\mathrm{D}^b(\Lambda) = \mathrm{D}^b(\mathbb{X})$ this finally induces an auto-equivalence of $\mathcal{H}$ fixing L. Moreover, φ is inner if and only if the induced functor on $\operatorname{mod}(\Lambda)$ is isomorphic to the identity. These constructions are mutually inverse. $\square$

3. The projective spectra and the Riemann sphere

Let $\mathbb{X}$ be a homogeneous exceptional curve over the real numbers. There are (up to duality) five cases of underlying tame bimodules M, namely $M = {}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$, ${}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}} \oplus {}_{\mathbb{R}}\mathbb{R}_{\mathbb{R}}$, ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}$, ${}_{\mathbb{H}}\mathbb{H}_{\mathbb{H}} \oplus {}_{\mathbb{H}}\mathbb{H}_{\mathbb{H}}$ or ${}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\overline{\mathbb{C}}}$, where in the last case $\mathbb{C}$ is acting on the second component via conjugation. In these cases we have $\operatorname{coh}(\mathbb{X}) \simeq \frac{\operatorname{mod}^{\mathbb{Z}}(R)}{\operatorname{mod}_0^{\mathbb{Z}}(R)}$, the quotient category modulo the Serre subcategory of $\mathbb{Z}$-graded modules of finite length, where R is one of the following $\mathbb{Z}$-graded algebras, respectively: $\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2)$, $\mathbb{R}[X,Y]$, $\mathbb{C}[X,Y]$, $\mathbb{H}[X,Y]$ or $\mathbb{C}[X,\overline{Y}]$, where here $Y\alpha = \overline{\alpha}Y$ for $\alpha \in \mathbb{C}$. In each case $\mathbb{X}$ is the projective spectrum of R. These "projective coordinate algebras" are graded factorial in the sense that each graded prime ideal in R of height one is generated by some homogeneous normal element, called prime element. Moreover, R is finitely generated as module over its center. In particular, each line bundle is up to isomorphism of the form $L(n)$, where $L(n)$ is the image of $R(n)$ (degree-shift by n) in the quotient category.

We describe the projective spectra: We list generators of the homogeneous prime ideals of height one, then the endomorphism skew-field of the corresponding simple sheaf and then the so-called symbol data (which we will need and explain below). (Compare also [4].) In the sequel, a point $x \in \mathbb{X}$ is called *real*, *complex* or *quaternion*, respectively, if the endomorphism ring

$\mathrm{End}(S_x)$ of the simple sheaf concentrated in x is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively. This defines the *colouring* on $\mathbb{X}$.

1.) $\mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2)=\mathbb{R}[x,y,z]$.

- $ax+by+cz \quad (a,b,c)\neq(0,0,0)$; $\mathbb{C}$; $\binom{1}{1}$.

Hence $\mathbb{X}$ can be identified with $\mathbb{S}^2/\pm 1$, the 2-sphere modulo antipodal points. This is homeomorphic to $\mathbb{P}_1(\mathbb{C})/\mathbb{Z}_2$, the Riemann sphere modulo the fixed-point free involution (given by $z\mapsto -1/\bar{z}$ on $\mathbb{P}_1(\mathbb{C})$). There are no real points.

2.) $\mathbb{R}[X,Y]$.

- X, $Y+\alpha X \quad \alpha\in\mathbb{R}$; $\mathbb{R}$; $\binom{1}{1}$.
- $(Y+zX)(Y+\bar{z}X) \quad z\in\mathbb{C}\setminus\mathbb{R}$; $\mathbb{C}$; $\binom{2}{2}$.

Hence $\mathbb{X}=\mathbb{P}_1(\mathbb{C})/\mathbb{Z}_2$ (identifying X, $Y+\alpha X$, $(Y+zX)(Y+\bar{z}X)$ with the class of ∞, α, z, respectively) where here $\mathbb{Z}_2$ is generated by the involution (given by $z\mapsto\bar{z}$) having fixed points (= real points). We have two regions, the boundary (= real points) having symbol data $\binom{1}{1}$ and the inner points are complex having symbol data $\binom{2}{2}$.

3.) $\mathbb{C}[X,Y]$.

- X, $Y+zX \quad z\in\mathbb{C}$; $\mathbb{C}$; $\binom{1}{1}$.

Here, $\mathbb{X}=\mathbb{P}_1(\mathbb{C})$, the Riemann sphere.

4.) $\mathbb{H}[X,Y]$.

- X, $Y+\alpha X \quad \alpha\in\mathbb{R}$; $\mathbb{H}$; $\binom{1}{1}$.
- $(Y+zX)(Y+\bar{z}X) \quad z\in\mathbb{C}\setminus\mathbb{R}$; $\mathbb{C}$; $\binom{2}{1}$.

Here $\mathbb{X}=\mathbb{P}_1(\mathbb{C})/\mathbb{Z}_2$ (as in case 2.), but the boundary is coloured quaternion.

5.) $\mathbb{C}[X,\overline{Y}]$.

- X, Y; $\mathbb{C}$; $\binom{1}{1}$
- $Y^2-\alpha X^2=(Y-\sqrt{\alpha}X)(Y+\sqrt{\alpha}X) \quad 0<\alpha\in\mathbb{R}$; $\mathbb{R}$; $\binom{2}{1}$
- $Y^2-\alpha X^2 \quad 0>\alpha\in\mathbb{R}$; $\mathbb{H}$; $\binom{2}{2}$
- $(Y^2-zX^2)(Y^2-\bar{z}X^2) \quad z\in\mathbb{C}\setminus\mathbb{R}$; $\mathbb{C}$; $\binom{4}{2}$.

In this case, the points of $\mathbb{X}$ are in ono-to-one correspondence with the elements of $\mathbb{P}_1(\mathbb{C})/\mathbb{Z}_2$ (mapping X, Y, $Y^2-\alpha X^2$ $(0\neq\alpha\in\mathbb{R})$, $(Y^2-zX^2)(Y^2-\bar{z}X^2)$ $(z\in\mathbb{C}\setminus\mathbb{R})$ to the class of ∞, 0, α, z in $\mathbb{P}_1(\mathbb{C})/\mathbb{Z}_2$, respectively). The boundary is coloured in a more interesting fashion as in the preceding cases and is indicated in Figure 1.

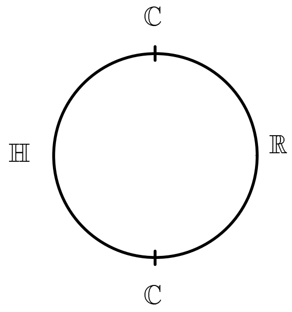


FIGURE 1

Let Σ be the Riemann sphere, so that each $\mathbb{X}$ is of the form Σ (in case 3.) or $\Sigma/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by an anti-automorphic involution κ which has fixed-points (in cases 2., 4. and 5. with the different colourings) forming the boundary or no fixed-points (in case 1.). In each case we define the group $\mathrm{Aut}'\,\mathbb{X}$ consisting of all conformal maps of the Riemann sphere Σ which in the cases different from case 3. commutes with κ and respects the colourings. Note that the group of conformal maps of Σ is given by Möbius transformations and the anti-automorphism $z \mapsto \bar{z}$, hence by $\mathrm{PGL}_2(\mathbb{C}) \rtimes \mathbb{Z}_2$ (see [6]). Then it is easy to see that in cases 1.–5. the group $\mathrm{Aut}'\,\mathbb{X}$ is given by, respectively, $\mathrm{SO}_3(\mathbb{R})$, $\mathrm{PGL}_2(\mathbb{R})$, $\mathrm{PGL}_2(\mathbb{C}) \rtimes \mathbb{Z}_2$, $\mathrm{PGL}_2(\mathbb{R})$ (see [6]) and $\mathbb{R}_+ \rtimes \mathbb{Z}_2$, where in the last case $\mathbb{R}_+$ is the set of diagonal matrices in $\mathrm{PGL}_2(\mathbb{R})$ with positive determinant (giving the Möbius transformations $z \mapsto \alpha z$ where $\alpha > 0$), and $\mathbb{Z}_2$ is generated by the inversion $I : z \mapsto 1/z$.

4. Action of automorphisms on $\mathbb{X}$

Let $\mathbb{X}$ be homogeneous. Each $\varphi \in \mathrm{Aut}\,\mathbb{X}$ permutes the points of $\mathbb{X}$. We show that we get in this way a natural surjective homomorphism of groups

$$\Phi : \mathrm{Aut}\,\mathbb{X} \longrightarrow \mathrm{Aut}'\,\mathbb{X}.$$

If $M = \mathbb{C} \oplus \mathbb{C}$, or $M = \mathbb{C} \oplus \overline{\mathbb{C}}$, then denote by γ the element in $\mathrm{Out}\,M$ induced by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$, and also the element in $\mathrm{Aut}\,\mathbb{X}$ (via the identification $\mathrm{Aut}\,\mathbb{X} = \mathrm{Out}\,M$) and call it *complex conjugation*. Obviously, $\gamma^2 = 1$, and $\Phi(\gamma)$ is the map induced by $z \mapsto \bar{z}$, which is the identity in case $M = \mathbb{C} \oplus \overline{\mathbb{C}}$; in this case we call γ *ghost-automorphism.*

Theorem 3. *Let $k = \mathbb{R}$ be the field of real numbers. Let $\mathbb{X}$ be a homogeneous exceptional curve. Φ is an isomorphism in the cases 1.–4., in the case 5. it is split surjective and has kernel generated by γ.*

Proof. We explicitly obtain $\mathrm{Aut}\,\mathbb{X}$ by calculating $\mathrm{Out}\,M$ for the underlying bimodule M. The cases $M = \mathbb{R} \oplus \mathbb{R}$ and $M = \mathbb{C} \oplus \mathbb{C}$ easily give $\mathrm{Out}\,M \simeq \mathrm{PGL}_2(\mathbb{R})$ and $\mathrm{Out}\,M \simeq \mathrm{PGL}_2(\mathbb{C}) \rtimes \langle\gamma\rangle$, respectively. Let $M = {}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$. For each $h \in \mathbb{H}^*$ denote by ι_h the inner automorphism given by $\iota_h(x) = h^{-1}xh$ for all $x \in \mathbb{H}$. Each $\varphi \in \mathrm{Aut}\,M$ has the form $\varphi = (1, \varphi, \iota_h)$, where $\varphi(x) = \varphi(1)h^{-1}xh$. We obtain a surjection $\mathbb{H}^* \rtimes \mathbb{H}^* \longrightarrow \mathrm{Aut}\,M$ with kernel $1 \rtimes \mathbb{R}^*$, hence $\operatorname{Aut} M \simeq \mathbb{H}^* \rtimes \mathbb{H}^*/\mathbb{R}^*$. Since every inner automorphism of the bimodule $\mathbb{H}$ is of the form $x \mapsto \alpha^{-1}xg$ for some $\alpha \in \mathbb{R}^*$, $g \in \mathbb{H}^*$, there is a surjection $\mathbb{R}^* \rtimes \mathbb{H}^* \longrightarrow \operatorname{Inn} M$ inducing an isomorphism $\operatorname{Inn} \mathbb{H} \simeq \mathbb{R}^* \rtimes \mathbb{H}^*/\mathbb{R}^*$. Hence $\operatorname{Out} M \simeq \mathbb{H}^*/\mathbb{R}^* \simeq \mathrm{SO}_3(\mathbb{R})$. By the correspondence of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k} \in \mathbb{H} = M$ (where $\mathbf{i}^2 = -1 = \mathbf{j}^2$, $\mathbf{k} = \mathbf{ij} = -\mathbf{ji}$) to x, y, z in the projective coordinate algebra (as described in [9, 4.3]), we get the isomorphism between $\operatorname{Aut} \mathbb{X}$ and $\operatorname{Aut}' \mathbb{X}$ in this case.

Let $M = \mathbb{H} \oplus \mathbb{H}$. Each element in $\operatorname{Aut} M$ is of the form $(\iota_h, \varphi, \iota_{h'})$. It is easy to see that

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto h^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} h'$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ uniquely determined as element of $\mathrm{PGL}_2(\mathbb{R})$ and h, $h' \in \mathbb{H}^*$ are all elements in $\operatorname{Aut} M$. Then $\operatorname{Out} M \simeq \mathrm{PGL}_2(\mathbb{R})$ follows immediately.

Finally, let $M = \mathbb{C} \oplus \overline{\mathbb{C}}$. It is easy to see that $\operatorname{Aut} M$ is generated by the subgroup U of matrices $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with a, $b \in \mathbb{C}^*$, and by $I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and by complex conjugation γ. Moreover, $\operatorname{Inn} M$ is given by the matrices $\begin{pmatrix} ab & 0 \\ 0 & a\bar{b} \end{pmatrix}$ with a, $b \in \mathbb{C}^*$. The surjective map $U \longrightarrow \mathbb{R}_+$, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} |a| & 0 \\ 0 & |b| \end{pmatrix}$ has kernel $\operatorname{Inn} M$, hence $\operatorname{Out} M \simeq (\mathbb{R}_+ \rtimes \langle I \rangle) \times \langle \gamma \rangle$.

In order to prove the theorem, one finally checks that each of the calculated automorphisms acts on the point set of $\mathbb{X}$ in the "natural" way, that is, the calculated matrices are mapped onto the associated Möbius transformations; only in case $M = \mathbb{C} \oplus \overline{\mathbb{C}}$ there is the exception that the element $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{R})$ with a, $b > 0$ yields the Möbius transformation $z \mapsto a^2 z/b^2$ giving a bijection from $\mathbb{R}_+$ onto itself. $\square$

As one surprising consequence of the theorem we see that in case $M = {}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$ the geometric structure on $\mathbb{X}$ also contains metric data. Namely, although $\mathbb{X}$ is topologically identical to the real projective plane $\mathbb{P}_2(\mathbb{R})$, geometrically $\mathbb{X}$ is different from $\mathbb{P}_2(\mathbb{R})$ equipped with the usual geometry, since this leads to the automorphism group $\mathrm{PGL}_3(\mathbb{R})$; but $\operatorname{Aut} \mathbb{X}$ consists just of those maps preserving the metric structure (angles).

5. The automorphism group of the derived category

Let $\mathbb{X}$ be an exceptional curve which is homogeneous with $\mathcal{H} = \operatorname{coh} \mathbb{X}$. Let M be the underlying tame bimodule of $\mathbb{X}$ and ε its *numerical type*, that is $\varepsilon = 1$ if M is a $(2, 2)$-bimodule and $\varepsilon = 2$ if M is a $(1, 4)$- or a $(4, 1)$-bimodule. For each $x \in \mathbb{X}$ let S_x be the simple sheaf concentrated in x, and let $f(x) = \frac{1}{\varepsilon} \cdot [\operatorname{Hom}(L, S_x) : \operatorname{End}(L)]$, $e(x) = [\operatorname{Hom}(L, S_x) : \operatorname{End}(S_x)]$, and $d(x) = f(x)e(x)$. We call $\begin{pmatrix} d(x) \\ f(x) \end{pmatrix}$ the *symbol data* of the point x.

We have listed above the symbol data in all cases over the real numbers. If we insert weights $p_1, \ldots, p_t$ into the pairwise distinct points $x_1, \ldots, x_t$ (called *exceptional points* or *weighted points*), then the Grothendieck group of $\overline{\mathcal{H}} = \mathcal{H}\begin{pmatrix} p_1 \ldots p_t \\ x_1 \ldots x_t \end{pmatrix}$ has the symbol

$$\left(\begin{array}{c|c} p_1, \ldots, p_t & \\ d_1, \ldots, d_t & \varepsilon \\ f_1, \ldots, f_t & \end{array} \right)$$

(compare [13]) where $f_i = f(x_i)$, $d_i = d(x_i)$ $(i = 1, \ldots, t)$. Recall that rows of the form $1, 1, \ldots, 1$ are omitted and ε only appears if $\varepsilon = 2$. Exceptional curves can be classified by their symbols. For example, $\mathbb{X}$ is domestic (tubular, wild, resp.) if the invariant

$$\sum_{i=1}^{t} d_i \Big(1 - \frac{1}{p_i}\Big) - \frac{2}{\varepsilon}$$

is < 0, $(= 0, > 0$, resp.). In particular, all domestic and tubular symbols can be listed ([12]).

Denote by $\sigma_i : \overline{\mathcal{H}} \longrightarrow \overline{\mathcal{H}}$ the shift-automorphism associated with the tube $\mathcal{U}_{x_i}$ (compare [14]) $(i = 1, \ldots, t)$. Moreover, let $x_0 \in \mathbb{X}$ be such that $e(x_0) = 1 = f(x_0)$, and denote by $\sigma_0 : \overline{\mathcal{H}} \longrightarrow \overline{\mathcal{H}}$ the induced shift-automorphism.

If $\mathbb{X}$ is an exceptional curve with sheaf category $\mathcal{H} = \operatorname{coh} \mathbb{X}$, then denote by $\operatorname{Pic} \mathbb{X}$ the subgroup of $\operatorname{Aut} \mathcal{H}$ which is generated by all shift-automorphisms; it is generated by $\sigma_0, \sigma_1, \ldots, \sigma_t$. Denote by $\operatorname{Pic}_0 \mathbb{X}$ the subgroup of $\operatorname{Pic} \mathbb{X}$ of shifts of degree zero. Denote by $\operatorname{Aut} \mathrm{D}^b(\mathbb{X})$ the group of isomorphism classes of auto-equivalences of the triangulated category $\mathrm{D}^b(\mathbb{X}) = \mathrm{D}^b(\mathcal{H})$.

Lemma 4. *Let $\mathbb{X}$ be an exceptional curve over the field of real numbers. Then* $\operatorname{Pic}(\mathbb{X})$ *is acting simply transitive on the set of isomorphism classes of line bundles.*

Proof. In the homogeneous case, by graded factoriality of the projective coordinate algebras as treated above, each shift-automorphism is naturally isomorphic to some degree-shift and each line bundle is a shift of L. Using the p-cycle construction in [13] it easily follows that we have natural isomorphisms $\sigma_i^{p_i} \simeq \sigma_0^{d_i}$ $(i = 1, \ldots, t)$, and then the assertion follows immediately. □

Theorem 5. *Let $\mathbb{X}$ be a tubular exceptional curve over the field $\mathbb{R}$. Then there is an exact sequence*

$$1 \longrightarrow \operatorname{Pic}_0 \mathbb{X} \rtimes \operatorname{Aut} \mathbb{X} \longrightarrow \operatorname{Aut} \mathrm{D}^b(\mathbb{X}) \longrightarrow V \longrightarrow 1,$$

where V is the braid group B_3 on three strands or a subgroup of B_3 of index 3. *More precisely, if s, l denote the generators of B_3 with defining relation $sls = lsl$, then $V = \langle l^n, s\rangle$ where n is either* 1 *or* 2.

Case	Symbol	Parameter	$\mathrm{Aut}\,\mathbb{X}_t$
D 1	$(p \mid 2)$	–	$\mathbb{R}/2\pi \rtimes \mathbb{Z}_2$
T 1	$(2\ 2 \mid 2)$	$t \in [0,\,1)$	$\mathbb{D}_4 \quad t = 0$ $\mathbb{V}_4 \quad t \neq 0$

TABLE 1. Domestic and tubular curves with $M = {}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$

Remark 6. (1) The group $\mathrm{Pic}_0\,\mathbb{X}$ coincides with the torsion group of the abelian group $\mathbb{L}(\mathbf{p},\mathbf{d})$ and is finite. A list of the occurring groups can be found in [11, Table 1], which also gives precise information about the occurring exponent n of the generator l.

(2) In $\langle l^2,\, s\rangle$ there is the defining relation $(l^2s)^2 = (sl^2)^2$.

Proof of (2). First observe that the subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ generated by the matrices $L^2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has defining relation $(L^2S)^2 = 1$. This can be proven by using a fundamental domain of this group (which can be found in [7]), using the method described in [17, 15.5]. Then we get the exact sequence

$$1 \longrightarrow \langle (l^2s)^2\rangle \longrightarrow \langle l^2,\, s \mid (l^2s)^2 = (sl^2)^2\rangle \longrightarrow \langle L^2,\, S\rangle \longrightarrow 1,$$

and the assertion follows as in [15]. □

Proof of the Theorem. As in [15] one has to prove split exactness of the following sequence

$$1 \longrightarrow \mathrm{Pic}\,\mathbb{X} \longrightarrow \mathrm{Aut}(\mathrm{coh}(\mathbb{X})) \longrightarrow \mathrm{Aut}\,\mathbb{X} \longrightarrow 1.$$

Here, the map $\mathrm{Aut}(\mathrm{coh}(\mathbb{X})) \longrightarrow \mathrm{Aut}\,\mathbb{X}$ is given by $F \mapsto \sigma \circ F$, where $\sigma \in \mathrm{Pic}\,\mathbb{X}$ is a shift-automorphism such that $\sigma F(L) = L$. This map is well-defined and surjective by the preceding lemma and clearly has kernel $\mathrm{Pic}\,\mathbb{X}$ and admits a section. The assertion now follows as in [15] using [11]. □

Remark 7. It follows from the split exact sequence in the preceding proof and from [15] that in the non-tubular case we have $\mathrm{Aut}\,\mathrm{D}^b(\mathbb{X}) = \mathbb{Z} \times (\mathrm{Pic}\,\mathbb{X} \rtimes \mathrm{Aut}\,\mathbb{X})$.

6. The domestic and tubular cases

If k is algebraically closed and of tubular weight type $(2\ 2\ 2\ 2)$, then $\mathbb{X}$ depends also on some parameter $\lambda \in k$, $\lambda \neq 0,\, 1$. More precisely, two such curves $\mathbb{X}(2\ 2\ 2\ 2;\lambda)$ and $\mathbb{X}(2\ 2\ 2\ 2;\mu)$ are isomorphic if and only if they have the same j-invariant $j(\lambda) = 2^8(\lambda^2-\lambda+1)^3/(\lambda^2(\lambda-1)^2)$ (see [15]). Moreover, also the automorphism group depends on this j-invariant [15]:

$$\mathrm{Aut}\,\mathbb{X} = \begin{cases} \mathbb{A}_4 & j = 0, \\ \mathbb{D}_4 & j = 1728, \\ \mathbb{V}_4 & j \neq 0,\, 1728. \end{cases}$$

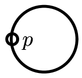
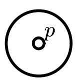
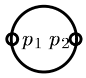
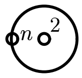
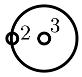
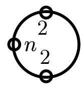
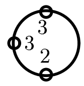
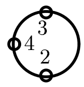
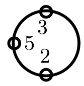
Case	Weights	Symbol	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{H}\oplus\mathbb{H}$
D 1		(p)	$\nabla\,\mathrm{PGL}_2(\mathbb{R})$	$\nabla\,\mathrm{PGL}_2(\mathbb{R})$
D 2		$\begin{pmatrix} p \\ 2 \end{pmatrix}$ $\begin{pmatrix} p \\ 2 \\ 2 \end{pmatrix}$	– $\mathbb{C}^*/\mathbb{R}^* \rtimes \mathbb{Z}_2$	$\mathbb{C}^*/\mathbb{R}^* \rtimes \mathbb{Z}_2$ –
D 3		$(p_1\ p_2)$	$\mathbb{R}^* \rtimes \mathbb{Z}_2 \quad p_1 = p_2$ $\mathbb{R}^* \quad p_1 \neq p_2$	$\mathbb{R}^* \rtimes \mathbb{Z}_2 \quad p_1 = p_2$ $\mathbb{R}^* \quad p_1 \neq p_2$
D 4		$\begin{pmatrix} 2 & n \\ 2 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & n \\ 2 & 1 \\ 2 & 1 \end{pmatrix}$	– $\mathbb{Z}_2$	$\mathbb{Z}_2$ –
D 5		$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$	– $\mathbb{Z}_2$	$\mathbb{Z}_2$ –
D 6		$(2\ 2\ n)$	$\mathbb{Z}_2 \quad n > 2$ $\mathbb{S}_3 \quad n = 2$	$\mathbb{Z}_2 \quad n > 2$ $\mathbb{S}_3 \quad n = 2$
D 7		$(2\ 3\ 3)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D 8		$(2\ 3\ 4)$	1	1
D 9		$(2\ 3\ 5)$	1	1

TABLE 2. Domestic curves with $M = \mathbb{R}\oplus\mathbb{R}$ and $M = \mathbb{H}\oplus\mathbb{H}$

Here, $\mathbb{A}_4$ denotes the alternating group (which is of order 12), $\mathbb{D}_4$ the dihedral group (of order 8) and $\mathbb{V}_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$ the Klein four group. In the other tubular cases (2 3 6), (2 4 4) and (3 3 3) and also in the domestic cases (p), $(p_1\ p_2)$, $(2\ 2\ n)$, (2 3 3), (2 3 4) and (2 3 5) there are no parameters since the group $\mathrm{PGL}_2(k)$ is acting strongly 3-transitive on $\mathbb{P}_1(k)$.

In this section we study for $k = \mathbb{R}$ in which domestic and tubular cases parameters occur and in which not. Moreover, we calculate the automorphism group in all these cases, which depends sometimes on the parameters. The results are given in the tables below. As a consequence we get

Corollary 8. (1) *There are no parameters in the domestic cases.*

(2) *If* $\mathbb{X}$ *is tubular then* $\mathrm{Aut}\,\mathbb{X}$ *is finite.*

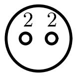
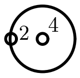
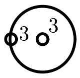
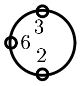
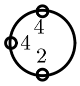
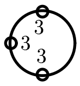
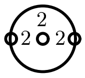
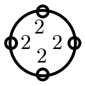
Case	Weights	Symbol	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{H}\oplus\mathbb{H}$
T 1		$\begin{pmatrix}2&2\\2&2\end{pmatrix}$ $\begin{pmatrix}2&2\\2&2\\2&2\end{pmatrix}$	– $\mathbb{V}_4$ $t\in(0,\,1)$	$\mathbb{V}_4$ $t\in(0,\,1)$ –
T 2		$\begin{pmatrix}2&4\\1&2\end{pmatrix}$ $\begin{pmatrix}2&4\\1&2\\1&2\end{pmatrix}$	– $\mathbb{Z}_2$	$\mathbb{Z}_2$ –
T 3		$\begin{pmatrix}3&3\\1&2\end{pmatrix}$ $\begin{pmatrix}3&3\\1&2\\1&2\end{pmatrix}$	– $\mathbb{Z}_2$	$\mathbb{Z}_2$ –
T 4		(2 3 6)	1	1
T 5		(2 4 4)	$\mathbb{Z}_2$	$\mathbb{Z}_2$
T 6		(3 3 3)	$\mathbb{S}_3$	$\mathbb{S}_3$
T 7		$\begin{pmatrix}2&2&2\\1&1&2\end{pmatrix}$ $\begin{pmatrix}2&2&2\\1&1&2\\1&1&2\end{pmatrix}$	– $\mathbb{Z}_2$ $t=\pi/2$; 1 $t\in(0,\,\pi)$ $t\neq\pi/2$	$\mathbb{Z}_2$ $t=\pi/2$; 1 $t\in(0,\,\pi)$ $t\neq\pi/2$ –
T 8		(2 2 2 2)	$\mathbb{A}_4$ $j=0$ $\mathbb{D}_4$ $j=1728$ $\mathbb{V}_4$ $j\neq 0,\,1728$	$\mathbb{A}_4$ $j=0$ $\mathbb{D}_4$ $j=1728$ $\mathbb{V}_4$ $j\neq 0,\,1728$

TABLE 3. Tubular curves for $M=\mathbb{R}\oplus\mathbb{R}$ and $M=\mathbb{H}\oplus\mathbb{H}$

We will not discuss the "classical" case $M=\mathbb{C}\oplus\mathbb{C}$ in the following. We also omit it in the tables. For this case we refer to [15]. But note, that we here consider automorphisms over $\mathbb{R}$, so that we have additionally complex conjugation.

6.1. **The cases with $M={}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}$.** If we take two distinct points on the 2-sphere (identifying antipodes) then obviously there is an automorphism mapping one point to the other. Hence there are no parameters in the case D 1 of Table 1. The automorphisms leaving one point $x\in\mathbb{S}^2$ fixed

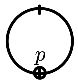
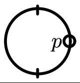
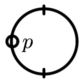
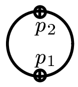
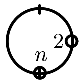
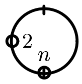
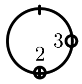
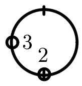
Case	Weights	Symbol	$\operatorname{Aut}\mathbb{X}$
D 1		(p)	$\mathbb{R}_+ \times \mathbb{Z}_2$
D 2		$\begin{pmatrix} p \\ 2 \end{pmatrix}$	$\mathbb{V}_4$
D 3		$\begin{pmatrix} p \\ 2 \\ 2 \end{pmatrix}$	$\mathbb{V}_4$
D 4		$(p_1\ p_2)$	$\mathbb{R}_+ \rtimes \mathbb{V}_4 \quad p_1 = p_2$ $\mathbb{R}_+ \times \mathbb{Z}_2 \quad p_1 \neq p_2$
D 5		$\begin{pmatrix} 2 & n \\ 2 & 1 \end{pmatrix}$	$\mathbb{Z}_2$
D 6		$\begin{pmatrix} 2 & n \\ 2 & 1 \\ 2 & 1 \end{pmatrix}$	$\mathbb{Z}_2$
D 7		$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$	$\mathbb{Z}_2$
D 8		$\begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$	$\mathbb{Z}_2$

TABLE 4. Domestic curves with $M = \mathbb{C} \oplus \overline{\mathbb{C}}$

are rotation around the axis through x and $-x$ by any angle and rotation around an orthogonal axis by angle π.

If we have two pairs of distinct points on the 2-sphere, then there is an automorphism mapping one pair to the other if and only if the cosine of their respective angles coincides; therefore we get a parameter $t \in [0, 1)$. The set of automorphisms leaving the set of two distinct points x, y fixed depends on the question whether these points are orthogonal or not. In the non-orthogonal case ($t \neq 0$) we have rotations around the axes $\mathbb{R}(x+y)$ and $\mathbb{R}(x-y)$ by angle π. In the orthogonal case there is additionally rotation by angle $\pi/2$ around the axis orthogonal to the x-y-plane. As result we get Table 1.

6.2. **The cases with $M = \mathbb{R} \oplus \mathbb{R}$ and $M = \mathbb{H} \oplus \mathbb{H}$.** Since each automorphism maps the boundary (= real points) onto itself we have to deal with a subgroup of $\mathrm{PGL}_2(\mathbb{R})$. Moreover, the group $\mathrm{PGL}_2(\mathbb{R})$ is acting 3-transitively on boundary points. Therefore there are no parameters in the cases where there are only boundary points, and at most three of them (cases D 1, D 3, D 6, D 7, D 8, D 9, T 4, T 5, T 6 and T 7 of Tables 2 and 3). (In Table 2 we denote by $\nabla\,\mathrm{PGL}_2(\mathbb{R})$ a group which is conjugate to the subgroup of

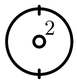
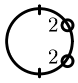
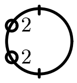
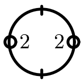
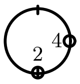
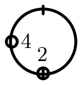
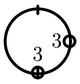
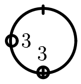
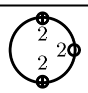
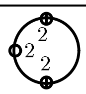
Case	Weights	Symbol	Parameter	$\operatorname{Aut}\mathbb{X}_t$
T 1		$\begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$	$t \in (0,\, \pi)$	$\mathbb{V}_4$
T 2		$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$	$t \in (0,\, 1)$	$\mathbb{V}_4$
T 3		$\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix}$	$t \in (0,\, 1)$	$\mathbb{V}_4$
T 4		$\begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 2 \end{pmatrix}$	$t \in (0,\, 1]$	$\mathbb{V}_4 \quad t = 1$ $\mathbb{Z}_2 \quad t \neq 1$
T 5		$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$	–	$\mathbb{Z}_2$
T 6		$\begin{pmatrix} 2 & 4 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$	–	$\mathbb{Z}_2$
T 7		$\begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$	–	$\mathbb{Z}_2$
T 8		$\begin{pmatrix} 3 & 3 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$	–	$\mathbb{Z}_2$
T 9		$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}$	–	$\mathbb{V}_4$
T 10		$\begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$	–	$\mathbb{V}_4$

TABLE 5. Tubular curves with $M = \mathbb{C} \oplus \overline{\mathbb{C}}$

$\mathrm{PGL}_2(\mathbb{R})$ formed by the upper triangular matrices.) In case D 6 the automorphism group depends on whether $n = 2$ or $n > 2$; in the first case it is the symmetric group $\mathbb{S}_3$, in the latter it is $\mathbb{Z}_2$.

In case D 2 of Table 2 it is easy to see that each inner point can be mapped to the imaginary unit i, hence there is no parameter. All automorphisms fixing i are given by $z \mapsto (az+b)/(-bz+a)$ (with $a,\, b \in \mathbb{R}$, not both zero), and one can compose these maps with the map $z \mapsto 1/z$ (mapping i to $-i$, which is identified with i.)

In the case (2 2 2 2) of four boundary points, there is a real parameter depending on the j-invariant as in the classical case.

In the cases where there is one boundary point and one inner point (D 4, D 5, T 2, T 3 of Tables 2 and 3) one sees that there is no parameter since any pair $(r,\, z)$ can be mapped to $(\infty,\, z')$, and any pair $(\infty,\, z)$ can be mapped to any (∞, z') (r real, z, z' complex). The automorphisms fixing ∞ and the imaginary unit i (up to sign) are the identity and the map $z \mapsto -z$.

In the case of two inner points (T 1 of Table 3) we have to check whether two pairs of distinct inner points lie in the same $\mathrm{PGL}_2(\mathbb{R})$-orbit or not. After applying suitable automorphisms, we can assume that the two pairs are (i, it) and (i, it'), where i is the imaginary unit and t, $t' > 0$, $\neq 1$. Then, these pairs lie in the same orbit if and only if $t = t'$, or $t = 1/t'$. Therefore we get a parameter t in the open interval $(0, 1)$. The subgroup of automorphisms fixing a pair (i, it) (as set) is generated by $z \mapsto -z$ and $z \mapsto -t/z$ (which permutes the two points), hence is isomorphic to the Klein four group $\mathbb{V}_4$.

6.3. **The cases with $M = \mathbb{C} \oplus \overline{\mathbb{C}}$.** For the notation of the weighted points in Tables 4 and 5 compare Figure 1. The calculations in this case are easy. We only discuss the cases where parameters do occur.

In the only case where inner points are involved (case T 1 in Table 5), if we have two inner points z and z', then there is an automorphism mapping z onto z' if and only if they are proportional (over $\mathbb{R}$). Therefore, we get a parameter $t \in (0, \pi)$, which is the angle of the polar coordinates (note that conjugates are identified). The automorphism group (fixing the inner point z) is generated by the ghost γ and by $|z|^2 \cdot I$.

In the cases T 2, T 3, T 4 in Table 5, let (r_1, r_2) and (r'_1, r'_2) be two pairs of (distinct) weighted points. For example, in the case T 2 we have positive real numbers. By stretching, we can assume that $r_1 = 1 = r'_1$, r_2, $r'_2 \neq 1$. Then (r_1, r_2) can be mapped to (r'_1, r'_2) if and only if $r_2 = r'_2$ or $r_2 = 1/r'_2$. Hence we get a parameter t in the open interval $(0, 1)$. The points 1 and r are fixed (as set) by the ghost γ and by $r \cdot I$. In the case T 4, if we also let $r_1 = 1$, then we have additionally the possibility $r_2 = -1$, therefore we get a parameter t in the half-open interval $(0, 1]$; the points r_1 and r_2 are fixed in this case only by γ, in case $t = 1$ ($r_2 = -1$) additionally by the inversion I.

It is easy to see that in the remaining cases of Tables 4 and 5 each pair (triple, singleton) of weighted points can be mapped into any other so that there is no parameter. Also the calculation of the automorphism group is straightforward.

Remark 9. There are some different tubular cases connected by derived equivalence due to the fact that there are tubular exceptional curves with two isomorphism classes of tubular families (which can be shown as in [10] using [11]): Each tubular curve in T 1 of Table 3 with underlying bimodule $\mathbb{R} \oplus \mathbb{R}$ is derived equivalent to one in T 3 of Table 5 and conversely. The same is true for the case T 1 of Table 3 with underlying bimodule $\mathbb{H} \oplus \mathbb{H}$ and T 2 of Table 5, for T 1 of Table 1 and T 4 of Table 5, for T 2 of Table 3 with underlying bimodule $\mathbb{R} \oplus \mathbb{R}$ and T 6 of Table 5, for T 2 of Table 3 with underlying bimodule $\mathbb{H} \oplus \mathbb{H}$ and T 5 of Table 5.

7. An example

In this final section we give an example of a tubular exceptional curve where the exact sequence in Theorem 5 splits.

Example 10. Denote by $\mathbb{C}$ and $\mathbb{C}'$ two different embeddings of the complex numbers into the skew-field $\mathbb{H}$ of quaternions: Denote by $\mathbf{i}$, $\mathbf{j}$ the generators with relations $\mathbf{i}^2 = -1 = \mathbf{j}^2$, $\mathbf{ji} = -\mathbf{ij}$. Then for example, we take $\mathbb{C} = \mathbb{R}\mathbf{1} \oplus \mathbb{R}\mathbf{i}$ and $\mathbb{C}' = \mathbb{R}\mathbf{1} \oplus \mathbb{R}\mathbf{i}'$, where $\mathbf{i}'$ is some pure quaternion of the form $\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{ji}$, $\mathbf{i}' \notin \mathbb{R}\mathbf{i}$, $(\alpha, \beta, \gamma) \in \mathbb{S}^2$. Let Λ be the tubular canonical $\mathbb{R}$-algebra given as tensor algebra of the species

$$\begin{tikzcd}
& \mathbb{C} \arrow[dr, "\mathbb{H}"] & \\
\mathbb{R} \arrow[ur, "\mathbb{C}"] \arrow[rr, "{}_{\mathbb{R}}\mathbb{H}_{\mathbb{H}}"] \arrow[dr, "\mathbb{C}'"'] & & \mathbb{H} \\
& \mathbb{C} \arrow[ur, "\mathbb{H}"'] &
\end{tikzcd}$$

modulo a certain ideal of relations (see [10]). We show

$$\operatorname{Aut} \mathrm{D}^b(\Lambda) \simeq \begin{cases} (\mathbb{Z}_2 \rtimes \mathbb{D}_4) \rtimes \langle l^2, s \rangle & \text{if } (\alpha, \beta, \gamma) \perp (1, 0, 0) \\ (\mathbb{Z}_2 \rtimes \mathbb{V}_4) \rtimes \langle l^2, s \rangle & \text{else.} \end{cases}$$

Denote by $\mathbb{X}$ the tubular curve associated with the central separating tubular family such that $\mathrm{D}^b(\mathbb{X}) \simeq \mathrm{D}^b(\Lambda)$ as triangulated categories [10]. More precisely, $\mathbb{X}$ arises from the projective spectrum of the $\mathbb{Z}$-graded algebra

$$\mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2) = \mathbb{R}[x, y, z]$$

by insertion of the weight 2 into the points x and $\alpha x + \beta y + \gamma z$. We have to show that the exact sequence from Theorem 5 splits: Let U be the subgroup of $\operatorname{Aut} \mathrm{D}^b(\mathbb{X})$ which is generated by the shift-automorphisms Φ_L, Φ_S associated to the tubes belonging to L and to some exceptional simple sheaf S, respectively. In order to show, that $\Phi_L \mapsto l^2$, $\Phi_S \mapsto s$ defines an isomorphism between U and $\langle l^2, s \rangle \subset B_3$, it is enough to see that we have the relation $\Phi_S \Phi_L \Phi_S \Phi_L \simeq \Phi_L \Phi_S \Phi_L \Phi_S$. Easy calculations show that this relation holds for the induced automorphisms of $\mathrm{K}_0\, \mathbb{X}$. With the arguments of [15, 7.1] and the explicit description of $\operatorname{Aut} \mathbb{X}$ it is enough to show that $\Phi_S \Phi_L \Phi_S \Phi_L \Phi_S^{-1} \Phi_L^{-1} \Phi_S^{-1} \Phi_L^{-1}$ lies in $\operatorname{Aut} \mathbb{X}$ and fixes all simple sheaves lying in homogeneous tubes. But this follows since $\Phi_L \Phi_S \Phi_L$ preserves the rank (up to sign) and Φ_S fixes homogeneous tubes.

References

1. W. W. Crawley-Boevey, *Regular modules for tame hereditary algebras*, Proc. London Math. Soc. **62** (1991), no. 3, 490–508.
2. V. Dlab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc., no. 173, American Mathematical Society, 1976.

3. ———, *Real subspaces of a quaternion vector space*, Canad. J. Math. **30** (1978), no. 6, 1228–1242.
4. V. B. Dlab, *An introduction to diagrammatical methods in representation theory*, Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, vol. 7, Universität Essen, 1981.
5. W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, Singularities, Representation of Algebras and Vector Bundles (Lambrecht 1985) (Berlin-Heidelberg-New York), Lecture Notes in Math., vol. 1273, Springer-Verlag, 1987, pp. 265–297.
6. G. A. Jones and D. Singerman, *Complex functions. An algebraic and geometric viewpoint*, Cambridge University Press, 1987.
7. N. Koblitz, *Introduction to elliptic curves and modular forms*, Graduate Texts in Mathematics, vol. 97, Springer-Verlag, Berlin-Heidelberg-New York, 1984.
8. D. Kussin, *Graduierte Faktorialität und die Parameterkurven tubularer Familien*, Ph.D. thesis, Universität Paderborn, 1997.
9. ———, *Factorial algebras, quaternions and preprojective algebras*, Algebras and Modules II (I. Reiten, S. O. Smalø, and Ø. Solberg, eds.), CMS Conf. Proc., vol. 24, Amer. Math. Soc., Providence, R. I., 1998, pp. 393–402.
10. ———, *Non-isomorphic derived-equivalent tubular curves and their associated tubular algebras*, J. Algebra **226** (2000), 436–450.
11. ———, *On the K-theory of tubular algebras*, Colloq. Math. **86** (2000), 137–152.
12. H. Lenzing, *A K-theoretic study of canonical algebras*, Representation Theory of Algebras (Cocoyoc, 1994) (R. Bautista, R. Martínez-Villa, and J. A. de la Peña, eds.), CMS Conf. Proc., vol. 18, Amer. Math. Soc., Providence, R. I., 1996, pp. 433–473.
13. ———, *Representations of finite dimensional algebras and singularity theory*, Trends in ring theory. Proceedings of a conference at Miskolc, Hungary, July 15–20, 1996 (V. Dlab et al., ed.), CMS Conf. Proc., vol. 22, Amer. Math. Soc., Providence, R. I., 1998, pp. 71–97.
14. H. Lenzing and J. A. de la Peña, *Concealed-canonical algebras and separating tubular families*, Proc. London Math. Soc. **78** (1999), no. 3, 513–540.
15. H. Lenzing and H. Meltzer, *The automorphism group of the derived category for a weighted projective line*, Comm. Algebra **28** (2000), no. 4, 1685–1700.
16. C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra **41** (1976), 269–302.
17. H. Zieschang, *Finite groups of mapping classes of surfaces*, Lecture Notes in Mathematics, vol. 875, Springer-Verlag, 1981.

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