Weighted projective lines and the tubular world

Lecture at ARTIG 3

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Paderborn, 1. Dezember 2023

A class of weighted projective curves arising in representation theory of finite dimensional algebras

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Introduction

By means of a suitably graded sheaf theory we introduce a new class of curves, called *weighted projective lines*, having an interpretation as lines in an appropriate *weighted projective space* $\mathbf{P}_n(\boldsymbol{p})$, with respect to a weight sequence $\boldsymbol{p} = (p_0, \ldots, p_n)$ of integers. We note that our approach to weighted projective spaces is similar to the treatment by Delorme [8], Dolgachev [10] and Beltrametti-Robbiano [6] but differs sensibly in spirit and content. Section 1 summarizes those results of joint investigation with D. Baer and P. Dowbor which are needed to put weighted projective lines into proper perspective; a complete account is under preparation. The main advantage of

Begin of Geigle-Lenzing paper 1987

our approach is that Serre's theorem (1.7) holds true, which removes all the pathologies ([6], Section 3) encountered in the former treatment of these spaces.

As becomes clear from the results of Sections 2 and 5, a weighted projective line $\boldsymbol{C}$ behaves like a smooth projective curve with respect to coherent sheaves and vector bundles on $\boldsymbol{C}$. This allows us to use all the methods familiar in this latter situation, see [24, 32, 1]. So the category of coherent sheaves $\mathrm{coh}(C)$ has Serre-duality (2.2), consequently almost-split sequences (2.3). Each coherent sheaf splits into a direct sum of a vector bundle and a torsion sheaf (2.4). By means of a Riemann-Roch theorem (2.9) we attach a (virtual) genus to $\boldsymbol{C}$, which is characteristic for the complexity of the classification problem for $\mathrm{coh}(C)$ (5.4).

To each sequence $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_n)$ of pairwise distinct elements of $\mathbf{P}_1(k)$, normalized such that $\lambda_0 = \infty$, $\lambda_1 = 0$, $\lambda_2 = 1$, we attach the two-dimensional subvariety $\boldsymbol{F}(\boldsymbol{p}, \boldsymbol{\lambda})$ of $\mathbf{A}_{n+1}$, given by the equations

$$X_i^{p_i} = X_1^{p_1} - \lambda_i X_0^{p_0}, \quad i = 2, \ldots, n. \tag{1.1.2}$$

$\boldsymbol{F}(\boldsymbol{p}, \boldsymbol{\lambda})$ is stable under the $G(\boldsymbol{p})$-action just described. Accordingly, the elements $f_i = X_i^{p_i} - X_1^{p_1} + \lambda_i X_0^{p_0}$, $(i = 2, \ldots, n)$ generate a homogeneous ideal $I(\boldsymbol{p}, \boldsymbol{\lambda})$ of $S(\boldsymbol{p})$. Hence

$$S(\boldsymbol{p}, \boldsymbol{\lambda}) = k[X_0, X_1, \ldots, X_n]/I(\boldsymbol{p}, \boldsymbol{\lambda}) = k[x_0, x_1, \ldots, x_n] \tag{1.1.3}$$

is again $\mathbf{L}(\boldsymbol{p})$-graded with $\deg(x_i) = \vec{x}_i$.

We are now going to endow the (set-theoretic) quotients $\mathbf{P}_n(\boldsymbol{p}) = k^{n+1} - \{0\}/G(\boldsymbol{p})$ and $\boldsymbol{C}(\boldsymbol{p}, \boldsymbol{\lambda}) = \boldsymbol{F}(\boldsymbol{p}, \boldsymbol{\lambda})/G(\boldsymbol{p})$ with an $\mathbf{L}(\boldsymbol{p})$-graded sheaf theory, defining on $\mathbf{P}_n(\boldsymbol{p})$ and $\boldsymbol{C}(\boldsymbol{p}, \boldsymbol{\lambda})$ the geometric structure of a weighted projective space, a weighted projective line, respectively.

Geigle-Lenzing paper (cont.)

Many incarnations of WPLs

- $S = \mathbb{S}(\mathbf{p}, \underline{\lambda}) = k[X_1, \ldots, X_n]/\big(X_i^{p_i} - X_1^{p_1} - \lambda_i X_2^{p_2} \mid i = 3, \ldots, n\big)$ and $\mathbb{L}(\mathbf{p})$-graded sheaf theory, Serre construction $\operatorname{coh}(\mathbb{X}) \;\simeq\; \operatorname{mod}^{\mathbb{L}(\mathbf{p})}(S)/\operatorname{mod}_0^{\mathbb{L}(\mathbf{p})}(S)$ (Geigle-Lenzing 1987)
- Inductive construction: insertion of weights
  - into prime elements in $k[X, Y]$, or
  - p-cycle construction (Lenzing 1996), parabolic structures (Seshadri)

  both akin to root stacks
- as Deligne-Mumford stacks (e.g. A. Takahashi 2008)
- as holomorphic 2-orbifolds, over $\mathbb{C}$ (Lenzing 2016)
- $\operatorname{vect}(\mathbb{X}) \;\simeq\; \mathrm{CM}^{\mathbb{L}(\mathbf{p})}(S)$
- Axiomatic description of category $\operatorname{coh}(\mathbb{X})$ (Lenzing 1996, Reiten-Van den Bergh 2000, K 2016)
- Ringel canonical algebras Λ and tilting theory $\mathcal{D}^b(\operatorname{coh}(\mathbb{X})) \simeq \mathcal{D}^b(\Lambda)$. Also Happel's theorem.

$\mathcal{D}^b(\operatorname{coh}(\mathbb{X})) \simeq \mathcal{D}^b(\Lambda)$ one of main motivations: study geometry of separating tubular families (cf. Lenzing-de la Peña PLMS 1999).

Advantages

- $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ is a hereditary abelian category. Canonical algebras typically have global dimension 2
- Serre duality: $\mathrm{Ext}^1(F, G) = \mathrm{D}\,\mathrm{Hom}(G, \tau F)$ with self-equivalence $\tau \colon \mathcal{H} \to \mathcal{H}$. More precisely: $\tau F = F(\vec{\omega})$.

  Replacing Auslander-Reiten formula.
- $\mathcal{H}$ has no non-zero projectives or injectives: Auslander-Reiten sequences exist for all indecomposable objects
- Correspondence: Λ canonical alg. $\overset{1:1}{\longleftrightarrow}$ $\mathbb{X}$ WPL

  works in both directions

Axiomatic categorical approach

Advantage: works very general, also in *non*commutative situations, e.g. noncommutative projective schemes à la Artin-Zhang

Philosophical background:

Theorem (Gabriel-Rosenberg reconstruction theorem)

Any scheme can be reconstructed uniquely up to isomorphism from the category of quasi-coherent sheaves on this scheme.

Noncommutative algebraic geometry: study of certain Grothendieck categories

Note: locally coherent Grothendieck category already uniquely determined by (abelian) subcategory of finitely presented (= coherent) objects

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 7, July 1997, Pages 1893–1901
S 0002-9939(97)04122-1

HEREDITARY NOETHERIAN CATEGORIES WITH A TILTING COMPLEX

HELMUT LENZING

(Communicated by Eric M. Friedlander)

In memory of Maurice Auslander

Abstract. We are characterizing the categories of coherent sheaves on a weighted projective line as the small hereditary noetherian categories without projectives and admitting a tilting complex. The paper is related to recent work with de la Peña (Math. Z., to appear) characterizing finite dimensional algebras with a sincere separating tubular family, and further gives a partial answer to a question of Happel, Reiten, Smalø (Mem. Amer. Math. Soc. **120** (1996), no. 575) regarding the characterization of hereditary categories with a tilting object.

A characterization of weighted projective lines

We are going to characterize the categories $\operatorname{coh}(\mathbb{X})$ of coherent sheaves on a weighted projective line $\mathbb{X}$ [3, 4].

Theorem 1. *Let k be an algebraically closed field. For a small connected abelian k-category $\mathcal{H}$ with finite dimensional morphism and extension spaces the following assertions are equivalent:*

(i) *$\mathcal{H}$ is equivalent to the category of coherent sheaves on a weighted projective line.*
(ii) *Each object of $\mathcal{H}$ is noetherian. $\mathcal{H}$ is hereditary, has no non-zero projectives, and admits a tilting complex.*
(iii) *Each object of $\mathcal{H}$ is noetherian, moreover*
   (a) *There exists an equivalence $\tau\colon \mathcal{H} \to \mathcal{H}$ (Auslander-Reiten translation) such that Serre duality $\operatorname{D}\operatorname{Ext}^1(A,B) \cong \operatorname{Hom}(B,\tau A)$ holds functorially in $A, B \in \mathcal{H}$.*
   (b) *The Grothendieck group $\operatorname{K}_0(\mathcal{H})$ is finitely generated free, and the Euler form $\langle -,- \rangle\colon \operatorname{K}_0(\mathcal{H}) \times \operatorname{K}_0(\mathcal{H}) \to \mathbb{Z}$ given on classes of objects of $\mathcal{H}$ by $\langle [X],[Y] \rangle = \dim_k \operatorname{Hom}(X,Y) - \dim_k \operatorname{Ext}^1(X,Y)$ is non-degenerate of determinant ± 1.*
   (c) *$\mathcal{H}$ has an object without self-extensions which is not of finite length.*

Lenzing's axioms 1996

Weighted noncommutative smooth projective curves

k a perfect field.

(NC 1) $\mathcal{H}$ is small, connected, abelian, and each object noetherian.

(NC 2) $\mathcal{H}$ is k-category with finite dimensional Hom- and Ext-spaces.

(NC 3) Serre duality $\operatorname{Ext}^1_{\mathcal{H}}(F,G) = \operatorname{D}\operatorname{Hom}_{\mathcal{H}}(G,\tau F)$ with self-equivalence τ.

(NC 4) $\mathcal{H}$ contains an object of infinite length.

Then objects of finite length $\mathcal{H}_0 = \coprod_{x\in\mathbb{X}} \mathcal{U}_x$ with $\mathcal{U}_x$ connected uniserial. $\mathbb{X}$ some index set.

(NC 5) $|\mathbb{X}| = \infty$.

Write: $\mathcal{H} = \operatorname{coh}(\mathbb{X})$. Call it: weighted n.c. smooth projective curve over k

Theorem

(NC 6) *$\forall x \in \mathbb{X}$ only $p(x) < \infty$ simples in $\mathcal{U}_x$. Almost all $p(x) = 1$.*

(NC 7) *$k(\mathcal{H})$ function (skew) field: $[k(\mathcal{H}) : Z(k(\mathcal{H}))] < \infty$ and $\exists$ unique smooth projective curve X with $Z(k(\mathcal{H})) = k(X)$.*

Function field

$\mathcal{H}_0$ = objects of finite length. Serre quotient

$\mathcal{H}/\mathcal{H}_0 \simeq \mathrm{mod}(D)$ for unique skew field D/k. Write $k(\mathcal{H}) := D$.

Perpendicular calculus (Geigle-Lenzing 1991): $\mathcal{H}$ has an underlying non-weighted curve $\mathcal{H}_{nw}$. Same function field.

weighted curve	$\mathcal{H} = \mathrm{coh}(\mathbb{X})$	$k(\mathcal{H})$
	$\vert$	
non-weighted curve	$\mathcal{H}_{nw} = \mathrm{coh}(\mathbb{X}_{nw})$	$k(\mathcal{H})$
	$\vert$	
centre curve	$\mathrm{coh}(X)$	$k(X)$

$\mathcal{H}$ (or $\mathbb{X}$) <u>commutative</u> :iff $k(\mathcal{H})$ commutative field. (Then $\mathcal{H}_{nw} = \mathrm{coh}(X)$.)

Euler characteristic

k perfect field = the categorical centre of $\mathcal{H}$.

$\chi'(\mathcal{H}) := \frac{1}{\bar{p}^2 s^2}\langle\langle L, L\rangle\rangle$ where L is the structure sheaf of $\mathcal{H}$, $\bar{p}$ = lcm of weights, $s^2 = [k(\mathcal{H}) : Z(k(\mathcal{H}))]$.

Theorem (K 2016)

(1)

$$\begin{aligned}\chi'_{orb}(\mathcal{H}) &= \chi'(X) - \frac{1}{2}\sum_x \Big(1 - \frac{1}{p(x)e_\tau(x)}\Big)[k(x):k] \\ &= \chi'(\mathcal{H}_{nw}) - \frac{1}{2}\sum_x \frac{1}{e_\tau(x)}\Big(1 - \frac{1}{p(x)}\Big)[k(x):k].\end{aligned}$$

(2) $\chi'_{orb}(\mathcal{H}\otimes \bar{k}) = \chi'_{orb}(\mathcal{H})$.

$p(x)$ = weight, $e_\tau(x)$ = order of functor τ on homogeneous tube $\mathcal{U}_x \subset \mathcal{H}_{nw}$, $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ for $x \in X$ closed.

Genus zero

Call $\mathcal{H}$ of genus zero :iff $\mathcal{H}$ admits a tilting bundle.

Then there is even a tilting bundle with endomorphism ring a canonical algebra in more general sense of Ringel–Crawley-Boevey. (Lenzing-de la Peña)

Let $\mathbf{e}$ be the finite vector of all $e_\tau(x) > 1$ counted $[k(x):k]$-times.

Proposition

$\mathcal{H}$ of genus zero $\Leftrightarrow$ $\chi'(\mathcal{H}_{nw}) > 0$ $\Leftrightarrow$

$\chi'(X) > 0$ & $\mathbf{e} = (), (e), (e_1, e_2), (2,2,e), (2,3,3), (2,3,4), (2,3,5)$.

Theorem

Let $k = \bar{k}$. Then $\mathcal{H}$ is of genus zero $\Leftrightarrow$ *$\mathbb{X}$ is a WPL.*

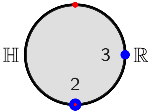
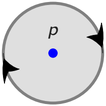
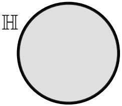
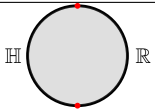
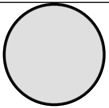
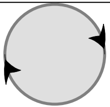
Proof.

"$\Rightarrow$" Tsen's theorem $\Rightarrow$ $k(\mathcal{H}) = k(X)$ commut. $\Rightarrow$ $\mathcal{H}_{nw} = \mathrm{coh}(X)$.

Genus zero $\Rightarrow$ $X \simeq \mathbb{P}^1_k$ $\Rightarrow$ $\exists p_1, \ldots, p_t > 1$: $\mathbb{X} = \mathbb{P}^1_k(p_1, \ldots, p_t)$. $\square$

Genus zero over $\mathbb{R}$

With categorical centre $\mathbb{R}$. (With centre $\mathbb{C}$ centre curve is $\mathbb{P}^1_{\mathbb{C}}$.)

$\mathcal{H}$	e.g.		
	s.t.	still $\chi'_{orb} > 0$	
$\mathcal{H}_{nw}$			
	$\mathbb{D}_{\mathbb{H}}$	$\mathbb{D}_{\mathbb{R},\mathbb{H}}$	
	$\text{Proj}(\mathbb{H}[X, Y])$	$\text{Proj}(\mathbb{C}[X, \overline{Y}])$	
centre			
	$\mathbb{D}$		$\mathbb{RP}^2 = \mathbb{S}^2/\pm$
	$\mathbb{P}^1_{\mathbb{R}} = \text{Proj}(\mathbb{R}[X, Y])$		$\text{Proj}\left(\frac{\mathbb{R}[X,Y,Z]}{X^2+Y^2+Z^2}\right)$

Noncomm. real Riemann-Hurwitz formula

$$\chi'_{orb}(\mathcal{H}) =$$
$$\chi'(\mathcal{H}_{nw}) - \frac{1}{4}\sum_{x}\left(1-\frac{1}{p(x)}\right) - \frac{1}{2}\sum_{y}\left(1-\frac{1}{p(y)}\right) - \sum_{z}\left(1-\frac{1}{p(z)}\right)$$

- x runs through the segmentation points •, i.e. with $e_\tau(x) = 2$.
- y through other boundary points
- z through inner points.

These segmentation points, separating real from quaternion segments, occur in even numbers on boundary cpts. by

Witt's theorem (1934) classifying finite central skew-field extensions of function fields $\mathbb{R}(X)$ with X real smooth proj. curve.

Representation type trichotomy

Theorem

Representation type of $\mathcal{H}$ (or $\mathrm{vect}(\mathbb{X})$) is

- *tame domestic* $\Leftrightarrow$ $\chi'_{orb}(\mathcal{H}) > 0$ $\Leftrightarrow$ *all indec. v.b. stable* *Fano case*
- *tubular* $\Leftrightarrow$ $\chi'_{orb}(\mathcal{H}) = 0$ $\Leftrightarrow$ *functor* $\tau \in \mathrm{Aut}(\mathcal{H})$ *is of finite order* *Calabi-Yau case* *(order of* $\tau = 1, 2, 3, 4$ *or* 6*)*
- *wild* $\Leftrightarrow$ $\chi'_{orb}(\mathcal{H}) < 0$ *general case*

In case $\chi'(\mathcal{H}) = 0$ we distinguish two cases

- $\mathcal{H}$ tubular $\bar{p} > 1$ (genus zero)
- $\mathcal{H}$ elliptic $\bar{p} = 1$ (no weight) (genus one)

In wider sense both are tubular.

Euler characteristic zero

Theorem

Let $\mathcal{H} = \mathrm{coh}(\mathbb{X})$ *with* $\chi'_{orb}(\mathcal{H}) = 0$.

(1) *Each indecomposable* E *in* $\mathcal{H}$ *is semistable of slope* $\mu(E) = \frac{\deg(E)}{\mathrm{rk}(E)}$.

(2) $\forall \alpha \in \mathbb{Q} \cup \{\infty\}$ *subcat.* $\mathbf{t}_\alpha$ *of semistable objects of slope* α *forms tubular family* $\neq 0$, *again parametrized by curve* $\mathbb{X}_\alpha$, *so that* $\mathcal{H}' = \mathrm{coh}(\mathbb{X}_\alpha)$ *is derived-equivalent to* $\mathcal{H}$ *and* $\chi'_{orb}(\mathcal{H}') = 0$.

(3) $\mathrm{Aut}(\mathcal{D}^b(\mathcal{H}))$ *acts on* $\{$*tubular families*$\}$ *with* ≤ 3 *orbits.*
($= 1$ *if* $k = \bar{k}$ *[Lenzing-Meltzer; Ringel];* ≤ 2 *if* $k = \mathbb{R}$*.)*

$k = \mathbb{R}$. Then 2 orbits lead to Fourier-Mukai partners $\mathcal{H}$ and $\mathcal{H}'$.

Weight-ramification-vector

$p(x) =$ weight, $e_\tau(x) =$ order of functor τ on homogeneous tube $\mathcal{U}_x \subset \mathcal{H}_{nw}$, $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ for $x \in X$ closed.

(WR-vector)

$\vec{w} := \big(p(x) \cdot e_\tau(x)\big)$ of all $p(x) \cdot e_\tau(x) > 1$, each repeated $[k(x) : k]$-times.

Proposition

$\chi'_{orb}(\mathcal{H}) = 0 \quad \Leftrightarrow$

- $g(X) = 0 \quad \& \quad \vec{w} = (2, 2, 2, 2), (3, 3, 3), (2, 4, 4), (2, 3, 6),$ *or*
- $g(X) = 1 \quad \& \quad \vec{w} = ().$

$\vec{w}$ is derived-invariant

In 2nd case $\mathbb{X} = X$ is (commutative) elliptic curve.

Integral Structures in Geometry and Representation Theory

Integral structures are found in many places throughout mathematics: as lattices in Euclidean space, as integral models of reductive groups and algebraic schemes, or as integral representations of groups and associative algebras. Even questions about the most basic example of an integral structure, the ring of integers $\mathbb{Z}$, quickly lead us into the fields of analysis, algebra, or geometry. In the same vein, investigations of integral structures are most successfully treated by viewing them from different perspectives, often requiring the usage of the most advanced mathematical methods and frequently leading to unexpected connections.

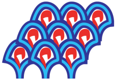
This point is illustrated by the classification of wallpaper groups, i.e., discrete groups of isometries of the plane that contain two linearly independent translations. As intricate double-periodic arabesques, we encounter them in the medieval Alhambra palace in Granada. It is a classical fact that there are precisely 17 wallpaper groups. This result has a geometric aspect, as it provides the number of flat compact orbifold surfaces. It also has an interpretation within representation theory: it is part of the classification of hereditary categories over the field of real numbers. The analogous classification of space groups in three-dimensional space is of importance in physics and chemistry.

As integral structures necessitate a combined approach from different mathematical sub-disciplines, we will embark on a broad research programme that underlines the unity of mathematics. Our endeavour ranges from algebraic geometry to analysis on manifolds, from geometric group theory and algebraic combinatorics to representation theory of associative algebras.

From the ARTIG website

The 17 wallpaper patterns

p1



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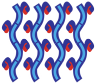
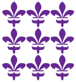


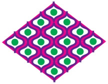
pgg

pm



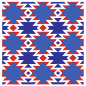
cm

cmm



pmg

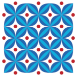
pmm



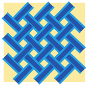
p2

p4

p4m

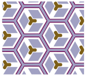


p4g



p3

p3m1



p31m

p6

p6m

Source: `https://mathworld.wolfram.com/WallpaperGroups.html`

Wallpaper patterns and tubular curves

Theorem

The 17 wallpaper patterns are in bijection with the elliptic or tubular curves $\mathcal{H}$ (up to parameter) over $\mathbb{R}$ whose function field $k(\mathcal{H})$ is commutative. (Next slide.)

Proof.

Well-known: wallpaper groups $\overset{1:1}{\leftrightarrow}$ flat real compact 2-orbifolds

$G \mapsto \mathbb{O} = E^2/G$ and $\mathbb{O} \mapsto (\widetilde{\mathbb{O}}, \pi_1^{orb}(\mathbb{O})) = (E^2, G)$. (Thurston, Montesinos, ...)

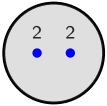
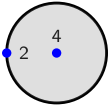
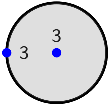
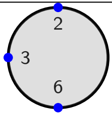
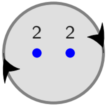
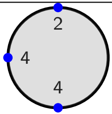
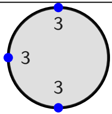
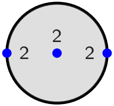
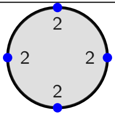
$H := G/\Gamma$ finite point group, Γ acting fixpoint-free (lattice)

$\Rightarrow$ $M := E^2/\Gamma$ compact Riemann surface, $M/H \simeq \mathbb{O}$

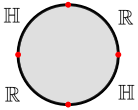
$\Rightarrow$ $\mathrm{coh}(\mathbb{O}) = \mathrm{coh}_H(M)$ hered. cat. with Serre duality ... of $\chi_{orb} = 0$

$\Rightarrow$ $\mathrm{coh}(\mathbb{O}) \simeq \mathrm{coh}(\mathbb{X})$, $\mathbb{X}$ tubular/elliptic curve, $k(\mathbb{X})$ commutative $\square$

The 17 wallpaper patterns (cont.)

$\mathbb{T}$ p1	$\mathbb{A}$ pm	$\mathbb{K}$ pg	$\mathbb{M}$ cm	
$\mathbb{P}^1_{\mathbb{C}}(2,2,2,2)$ $\mathbb{S}2222$ p2	$\mathbb{P}^1_{\mathbb{C}}(3,3,3)$ $\mathbb{S}333$ p3	$\mathbb{P}^1_{\mathbb{C}}(2,4,4)$ $\mathbb{S}442$ p4	$\mathbb{P}^1_{\mathbb{C}}(2,3,6)$ $\mathbb{S}632$ p6	
 $\mathbb{D}22$ pmg	 $\mathbb{D}4\bar{2}$ p4g	 $\mathbb{D}3\bar{3}$ p31m	 $\mathbb{D}\bar{6}\bar{3}\bar{2}$ p6m	 $\mathbb{P}22$ pgg
 $\mathbb{D}\bar{4}\bar{4}\bar{2}$ p4m	 $\mathbb{D}\bar{3}\bar{3}\bar{3}$ p3m1	 $\mathbb{D}2\bar{2}\bar{2}$ pmm	 $\mathbb{D}\bar{2}\bar{2}\bar{2}\bar{2}$ cmm	

An interesting non-commutative elliptic curve



$\mathbb{D}_{\mathbb{R},\mathbb{H},\mathbb{R},\mathbb{H}}$ Weights: (). $\vec{w} = (2, 2, 2, 2)$

Proposition (K 2016)

- *$\mathbb{D}_{\mathbb{R},\mathbb{H},\mathbb{R},\mathbb{H}}$ has no Fourier-Mukai partner.*
- *The stable bundles of degree 0 again parametrized by $\mathbb{D}_{\mathbb{R},\mathbb{H},\mathbb{R},\mathbb{H}}$.*
- *The line bundles of degree 0 are in bijection with the non-quaternion boundary points.*
- *the structure sheaf L corresponds to one of the segmentation points.*

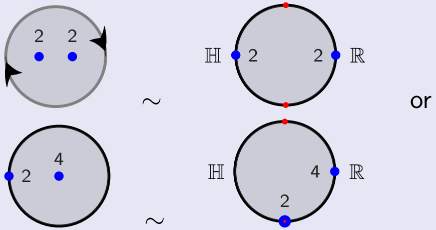
Additional non-commutative tubular curves

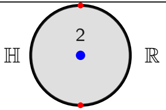
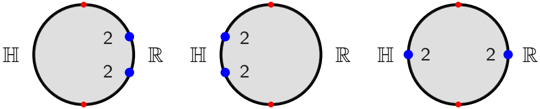
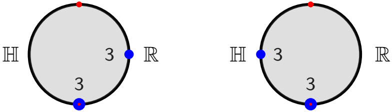
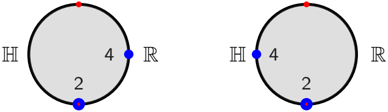
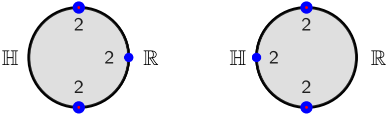
Over $\mathbb{R}$ there are 22 cases with $\chi'_{orb}(\mathcal{H}) = 0$ and $k(\mathcal{H})$ not commutative:

- Elliptic: $\mathbb{A}_{\mathbb{R},\mathbb{H}}$, $\mathbb{A}_{\mathbb{H},\mathbb{H}}$, $\mathbb{M}_{\mathbb{H}}$ and $\mathbb{D}_{\mathbb{R},\mathbb{H},\mathbb{R},\mathbb{H}}$
- Tubular:
  - The 8 cases from above with centre curve $\mathbb{D}$ replaced by $\mathbb{D}_{\mathbb{H}}$
  - 10 cases on next slide

(Derived-equivalences)

- $\mathbb{K} \quad \sim \quad \mathbb{A}_{\mathbb{R},\mathbb{H}}$ (K 2016; J. Rosenberg 2017)
- $+$ 5 tubular cases (K 1998), e.g.



$\mathcal{H}$	weights	$\vec{w}$
	(2)	$(2,2,2,2)$
	$(2,2)$	$(2,2,2,2)$
	$(3,3)$	$(2,3,6)$
	$(2,4)$	$(2,4,4)$
	$(2,2,2)$	$(2,4,4)$

Quasicoherent sheaves

- $\vec{\mathcal{H}} = \mathrm{Qcoh}(\mathbb{X}) := \mathrm{Lex}(\mathcal{H}^{op}, \mathrm{Ab})$ hereditary, locally noetherian Grothendieck category
- $\mathcal{H} = \mathrm{fp}\,\vec{\mathcal{H}}$
- $\mathcal{M}(r) := {}^{\perp_0}\,\mathrm{vect}\,\mathbb{X} \quad = \text{objects of slope } \infty$
- $\mathbf{L}$ tilting object given by $(\mathrm{vect}\,\mathbb{X})^{\perp_1} = \mathbf{L}^{\perp_1}$ "Lukas tilt. obj."

Tilting-cotilting-correspondence

$\mathbb{X}$ any weight type, any χ'_{orb}.

Proposition (Angeleri-Hügel-K, Laking-K)

$^{\perp_1}T \cap \mathcal{H} = {}^{\perp_1}C \cap \mathcal{H}$ *induces bij. corresp.* $T \mapsto \Gamma(T) = C$ *between*

- *tilting* T *of finite type up to* Add-*equivalence* $\sim$
- *cotilting* C *up to* Prod-*equivalence* $\sim$

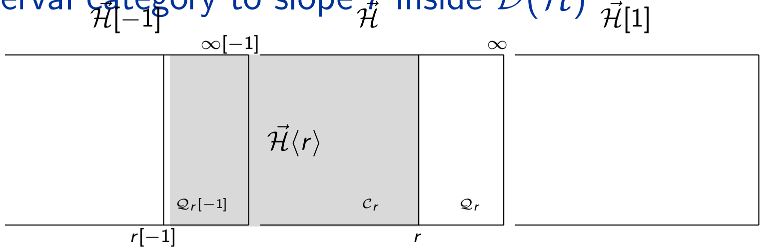
Theorem (K 2023)

- T *tilting of slope* ∞. *Then*

$$\Gamma(T) \sim \mathrm{PE}(T) \oplus \mathcal{K}.$$

- $\mathbf{L}$ *Lukas tilting of slope* ∞. *Then* $\mathrm{PE}(\mathbf{L}) \oplus \mathrm{PE}(\mathbf{L})/\mathbf{L}$ *is* $\sim$ *cotilting object given by indecomposable summands*
  - *adics* $\widehat{S}$ *for each simple object* S
  - *generic sheaf* $\mathcal{K}$ *of slope* ∞.

Interval category to slope r inside $\mathcal{D}(\vec{\mathcal{H}})$



- $\mathcal{D}(\vec{\mathcal{H}}\langle r\rangle) = \mathcal{D}(\vec{\mathcal{H}})$
- $\operatorname{fp}\vec{\mathcal{H}}\langle r\rangle = \mathcal{H}\langle r\rangle := \bigvee_{\beta>r} \mathbf{t}_\beta[-1] \vee \bigvee_{\alpha<r} \mathbf{t}_\alpha$ (is abelian)
- $\mathcal{H}\langle r\rangle$ satisfies Serre duality
- $\vec{\mathcal{H}}\langle r\rangle$ always locally coherent
  locally noetherian $\Leftrightarrow$ r rational (or ∞)
- all simple objects in $\vec{\mathcal{H}}\langle r\rangle$ are in definable $\mathcal{M}(r) := {}^{\perp_0}\mathcal{H}\langle r\rangle =:$ objects of slope r.

Injective cogenerator

r irrational

$\mathbf{W} := \mathrm{E}(\bigoplus_S \mathrm{E}(S))$ where S runs through simples/iso
is minimal injective cogenerator of $\vec{\mathcal{H}}\langle r\rangle$.

Proposition

- $\mathbf{W} \in \mathcal{M}(r)$
- $\mathbf{W}$ *<u>not</u>* Σ*-(pure-)injective*
- $E \in \vec{\mathcal{H}}$ *(or* $\vec{\mathcal{H}}\langle r\rangle$*) indecomposable pure-injective of slope* r
  $\Leftrightarrow \quad E \in \mathrm{Prod}(\mathbf{W})$ *indecomposable*

Lukas tilting object of irrational slope

Theorem (Angeleri Hügel-K 2017)

r irrational. $\exists$ "unique"

- *tilting object $\mathbf{L}$ in $\vec{\mathcal{H}}\langle r\rangle$ of slope r (generalized Lukas tilting object)*

$\mathbf{L}$ is projective generator of $\mathcal{M}(r)$

Theorem (K 2023)

For (sufficiently large) set I for $T := \mathbf{L}^{(I)}$ the minimal (pure-) injective resolution

$$0 \to T \to \mathrm{E}(T) \to \mathrm{E}(T)/T \to 0$$

in $\vec{\mathcal{H}}\langle r\rangle$ satisfies

- *$\mathbf{W}$ is direct summand of $\mathrm{E}(T)/T$, hence $\mathrm{Prod}(\mathbf{W}) = \mathrm{Prod}(\mathrm{E}(T)/T)$*
- *for each simple object S the injective envelope $\mathrm{E}(S)$ is not direct summand of $\mathrm{E}(T)$. "$\mathrm{E}(T)$ torsionfree"*

The Ziegler spectrum at irrational slope

Theorem (K 2023)

Let $\mathscr{S} \subset \mathcal{H}\langle r\rangle$ be a Serre subcategory. If $\mathscr{S}$ contains an indecomposable object in one homogeneous tube, then $\mathscr{S} = \mathcal{H}\langle r\rangle$.

Proposition

There are bijective correspondences between

- *Serre subcategories of $\mathcal{H}\langle r\rangle$*
- *definable subcategories of the exactly definable category $\mathcal{M}(r)$ of objects of slope r*
- *the closed subsets of $\mathrm{ZSp}(\mathcal{M}(r)) = \mathrm{Sp}(\vec{\mathcal{H}}\langle r\rangle)$*

Corollary

If $\mathcal{H}$ is elliptic, then $\mathrm{Sp}(\vec{\mathcal{H}}\langle r\rangle)$ has the indiscrete topology.

Hefei 3/2016

Thank you!